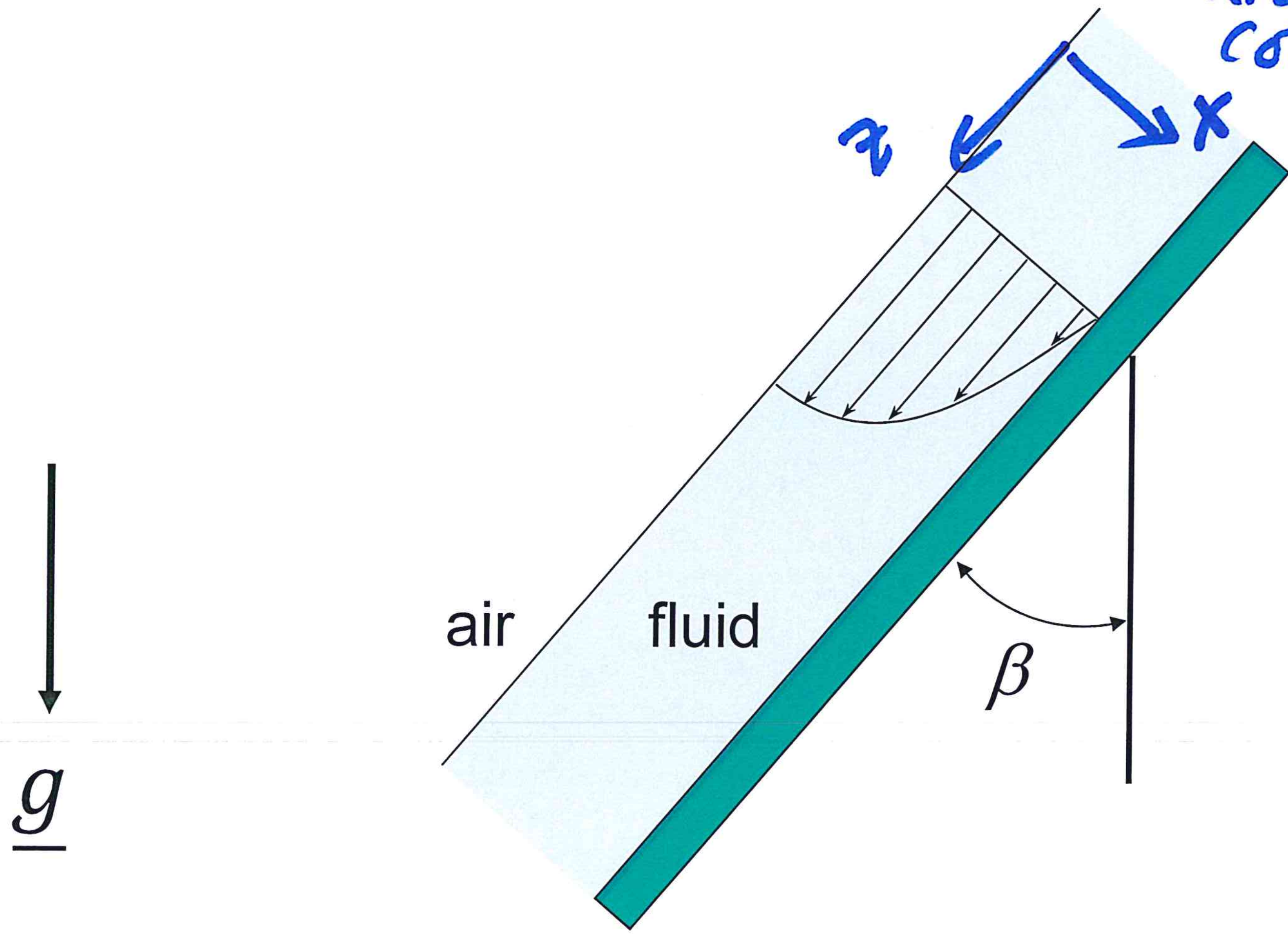
Flow Down Incline — w/ Navier-Stokes and continuity eqn. (1)

18 Sept 19



$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz}$$

2

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

0

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

conclude:

$$\boxed{\frac{\partial v_z}{\partial z} = 0}$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

ρ constant

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z}\right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z}\right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi}\cot\theta}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta\phi}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta}\cot\theta}{r}\right) + \rho g_\phi$$

$v_x = 0$ $v_y = 0$ 3

$\rho g \sin\beta$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x\frac{\partial v_x}{\partial x}} + \cancel{v_y\frac{\partial v_x}{\partial y}} + \cancel{v_z\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \cancel{\frac{\partial^2 v_x}{\partial y^2}} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right) + \rho g_x$$

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x\frac{\partial v_y}{\partial x}} + \cancel{v_y\frac{\partial v_y}{\partial y}} + \cancel{v_z\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right) + \cancel{\rho g_y} \quad 0$$

steady 0

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x\frac{\partial v_z}{\partial x}} + \cancel{v_y\frac{\partial v_z}{\partial y}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right) + \cancel{\rho g_z} \quad \rho g\cos\beta$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right.$$
$$\left. - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right.$$
$$\left. + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right.$$
$$\left. + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) + \rho g_\phi$$

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

EOM = NS = Micro Momentum Bal (4)

(fluid statics)

x-component:

$$0 = -\frac{\partial P}{\partial x} + \rho g \sin\beta$$

y-component

$$0 = -\frac{\partial P}{\partial y}$$

z-component

$$0 = -\cancel{\frac{\partial P}{\partial z}} + \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g \cos\beta$$

SOLVE

modeling assumption

($P = P_{atm}$ at $x = 0$; fluid statics = same level, same P)

$$-\frac{\rho g \cos\beta}{\mu} = \frac{d^2 v_z}{dx^2}$$

$v_z(x)$ only

(we said wide flow; continuity said $\frac{\partial v_z}{\partial z} = 0$)

to integrate:

let $$\Phi \equiv \frac{dv_z}{dx}$$

$$\frac{d\Phi}{dx} = \frac{d^2 v_z}{dx^2}$$

$$\frac{d\Phi}{dx} = \left(\frac{-\rho g \cos\beta}{\mu}\right)$$

integrate:

$$\frac{dv_z}{dx} = \Phi = \left(\frac{-\rho g \cos\beta}{\mu}\right)x + C_1$$

$$\frac{dV_z}{dx} = \left(\frac{-\rho g \cos\beta}{\mu}\right)x + C_1$$

$$\boxed{V_z = \left(\frac{-\rho g \cos\beta}{\mu}\right)\frac{x^2}{2} + C_1 x + C_2}$$

BC: $x = 0$ $\quad \frac{dV_z}{dx} = 0$ (stress matching w/ AIR)

$x = H$ $\quad V_z = 0$ (no slip)

<u>Solve</u>. (as before)

(7)

Engineering Quantities of Interest

average velocity

$$\langle v_z \rangle \equiv \frac{\int_0^W \int_0^H v_z \, dx \, dy}{\int_0^W \int_0^H dx \, dy}$$

H is the height of the film; W is the width

volumetric flow rate

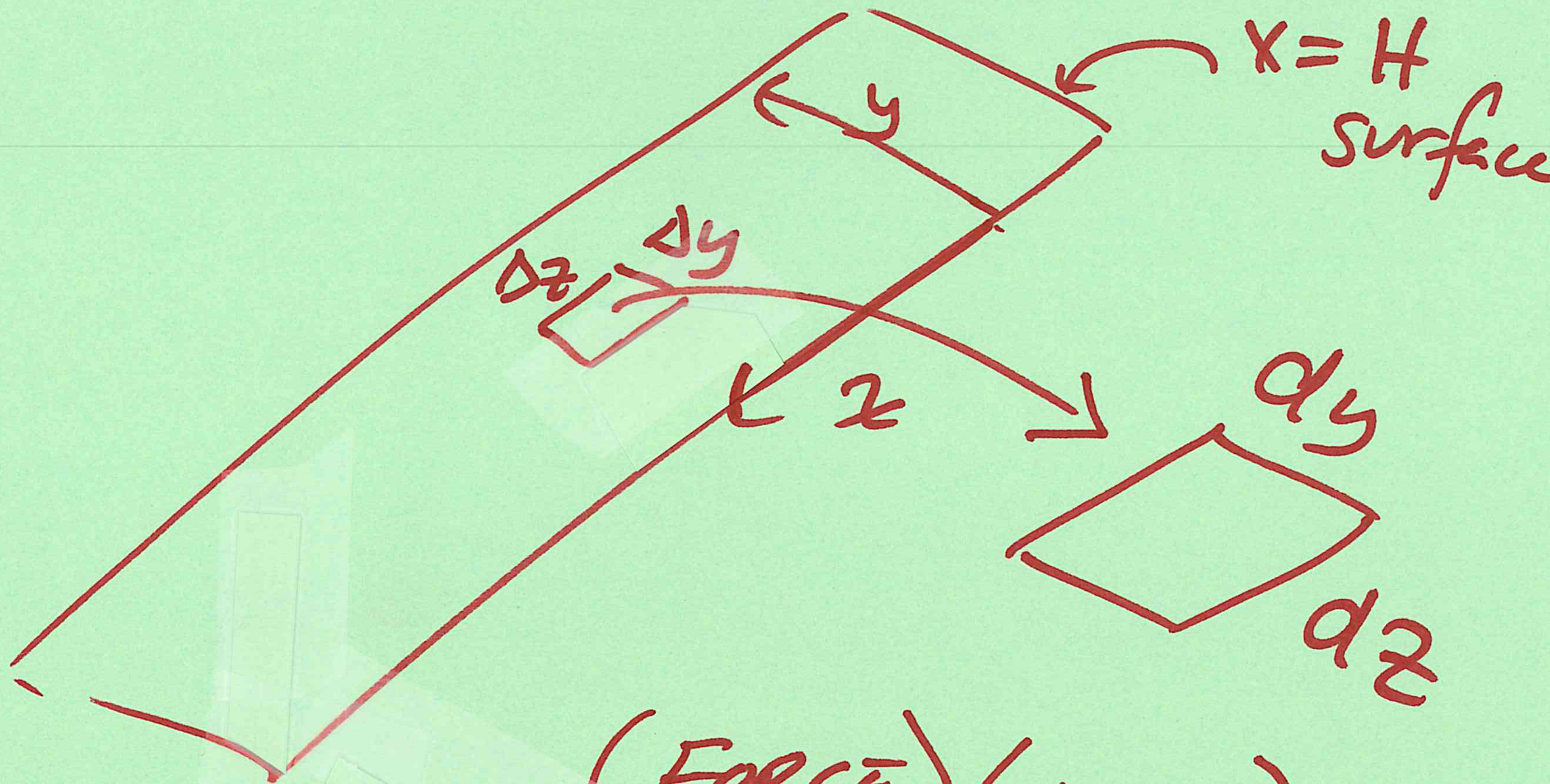
$$Q = \int_0^W \int_0^H v_z \, dx \, dy = WH \langle v_z \rangle$$

z-component of force on the wall

$$F_z = \int_0^L \int_0^W \tilde{\tau}_{xz}\Big|_{x=H} \, dy \, dz$$

where does this come from?

(The expressions are different in different coordinate systems)



$$\left(\frac{\text{FORCE}}{\text{AREA}}\right)(\text{AREA})$$

$$\underbrace{\int_0^L \int_0^W}_{\text{ADD IT UP}} \tau_{xz}\Big|_{x=H} \, dy\, dz$$

$0 < y \leq W$
$0 < z \leq L$

FORCE $\displaystyle\int_0^L \int_0^W \underbrace{\tilde{\tau}_{xy}\Big|_{x=H}} \, dy\, dz$

Newton's Law of Viscosity: $\displaystyle \mu \underbrace{\frac{dv_z}{dx}\Big|_{x=H}}$

from $v_z(x)$

$$\frac{\rho g \cos\beta}{2\mu}(-2x)\Big|_{x=H}$$

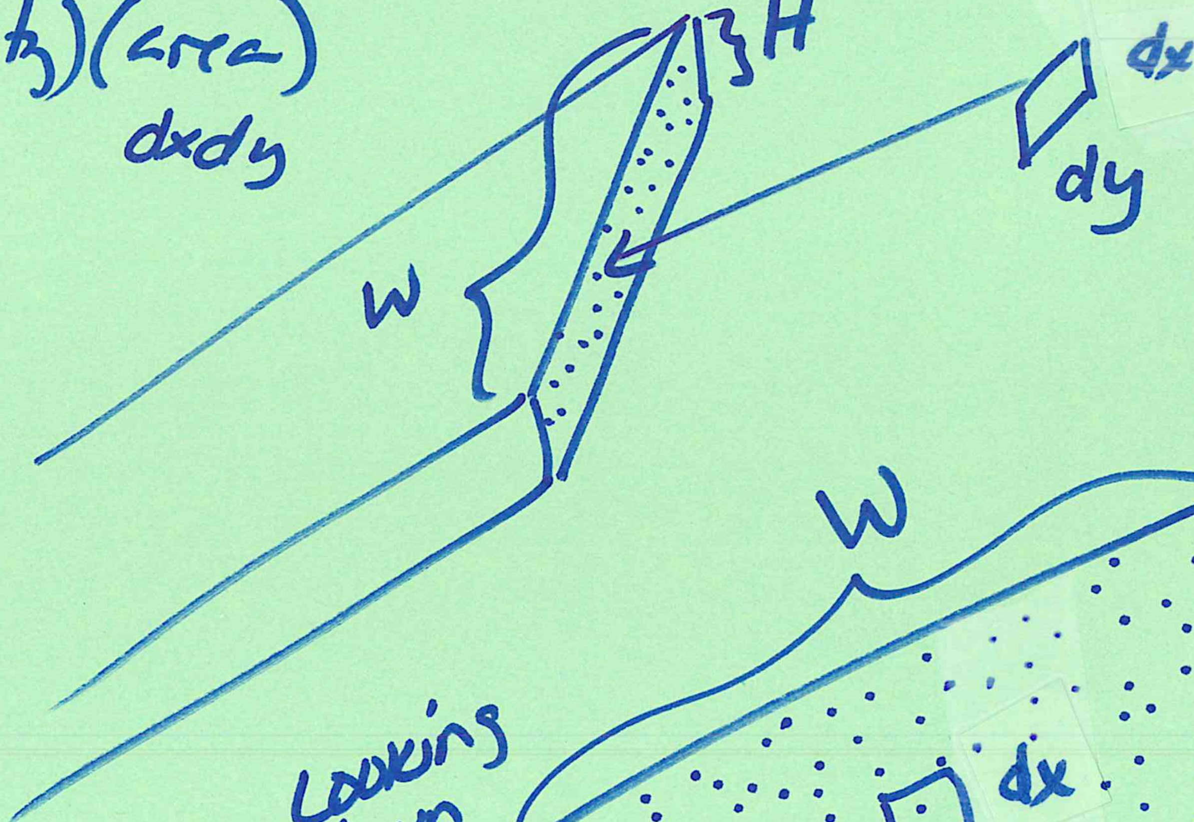
$$\int_0^L \int_0^W \frac{\cancel{\mu}\rho g \cos\beta}{\cancel{2\mu}}(-\cancel{2})H \, dy\, dz$$

$$\boxed{\text{FORCE} = \rho g \cos\beta \, (\neg) H W L}$$

Where does flow rate Q eqn come from?

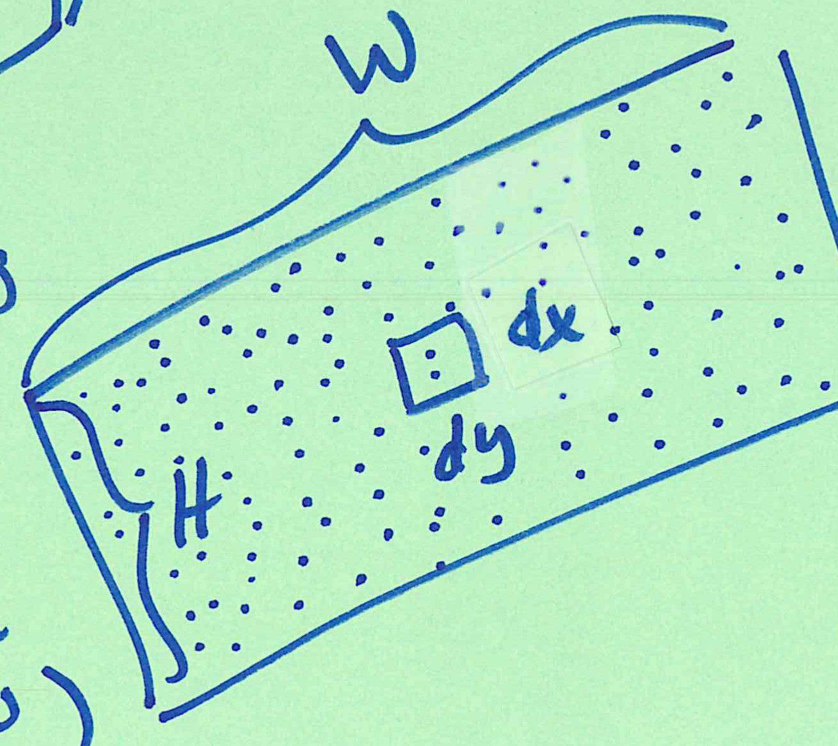
$$Q = (\text{velocity})(\text{area})$$

$$\underbrace{\iint}_{\text{add it up over entire cross section}} v_z \, dx\,dy$$



Looking down stream:

(cross-section of flow)



add up across the cross-section

velocity, crossectional area

$$Q = \int_0^W \int_0^H \underbrace{v_z}_{} \, dx\,dy$$

$$= \int_0^W \int_0^H \frac{\rho g \cos\beta}{2\mu}\left(H^2 - x^2\right) dx\,dy$$

$$= \frac{\rho g \cos\beta}{2\mu} W \underbrace{\int_0^H \left(H^2 - x^2\right) dx}_{\left(H^2 x - \frac{x^3}{3}\right)\Big|_0^H}$$

$$H^3 - \frac{H^3}{3} = \frac{2H^3}{3}$$

$$\boxed{Q = \frac{\rho g \cos\beta\, W}{\cancel{2}\mu}\,\frac{H^3(\cancel{2})}{3}}$$

12

13

Engineering Quantities of Interest

(any flow)

volumetric flow rate

$$\boxed{Q = \iint_S (\hat{n} \cdot \underline{v})\, dS}$$

average velocity

$$\boxed{\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v})\, dS}{\iint_S dA} = \frac{Q}{S}}$$

z-component of force on the wall

$$\boxed{F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}\right)\right]_{surface} dS}$$

*For more complex flows, we use the **Gibbs notation** versions (will discuss soon).*