1. Quick Start B): Using Calculus in Fluid Mechanics

1. Calculate flow rate
2. Calculate average velocity
3. Express velocity in a fluid as a vector field
4. Express forces on surfaces due to fluids
5. Express torques on surfaces due to fluids

1.1: Flow rate: $Q$ or $\dot{V}$

**General:**

$$Q = \iint_{\text{area}} (\mathbf{v} \cdot \hat{n}) d(area)$$

**Tube flow:**

$$Q = \int_0^{2\pi} \int_0^R v_z(r) r dr d\theta$$

$(\mathbf{v} \cdot \hat{n})$ is the component of $\mathbf{v}$ in the direction normal to the area
Common surface shapes:

- **rectangular**: \( d(\text{area}) = dx \, dy \)
- **circular**: \( d(\text{area}) = r \, dr \, d\theta \)
- **surface of cylinder**: \( d(\text{area}) = R \, d\theta \, dz \)
- **spherical**: \( d(\text{area}) = (r \, d\theta) (r \sin \theta \, d\phi) = r^2 \sin \theta \, d\theta \, d\phi \)

1.2: Average velocity: \( \langle v \rangle \)

General: \[ \langle v \rangle = \frac{Q}{\text{area}} \]

Tube flow: \[ \langle v \rangle = \frac{Q}{\pi R^2} \]

“area” is the cross-sectional area normal to flow
Example: In turbulent flow, it is often a good assumption to assume that the velocity profile is completely flat, that is, the velocity does not vary across the cross section. Using this assumption, what is the flow rate in turbulent flow in a tube of radius $R$?

$$\text{velocity} = \text{constant} = V$$

Example 1.12: The shape of the velocity profile for a steady flow in a tube is found to be given by $f(r)$ below. Over the range $0 < r < 10$ mm, ($R=10$mm), what is the average value of the velocity?

$$\frac{v_z}{v_{\text{max}}} = f(r) = 1 - \left(\frac{r}{10}\right)^2$$
1.3: Vectors

\[ \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{xyz} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} = \begin{pmatrix} v_r \\ v_\theta \\ v_\phi \end{pmatrix}_{r\theta\phi} \]

same vector, different coordinate systems

\[ |\mathbf{v}| = \mathbf{v} = \text{vector magnitude} \]

\[ \frac{\mathbf{v}}{v} = \hat{v} = \text{unit vector} \]

We will choose coordinate systems for convenience.

Vector plot of the velocity field in creeping flow around a sphere

The flow is a steady upward flow; the length and direction of the vector indicates the velocity at that location.
Vectors – Cartesian coordinate system

\[ \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x \hat{e}_x + v_y \hat{e}_y + v_z \hat{e}_z \]

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3 \]

\[ \mathbf{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \]

- We do algebra with these the same way as with other quantities
- The cartesian basis vectors are constant (with position)

Vectors – Cylindrical coordinate system

\[ \mathbf{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \]

- The cylindrical basis vectors are variable (with position)

\[ \begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*} \]

\[ \begin{align*}
\hat{e}_r &= \cos \theta \hat{e}_x + \sin \theta \hat{e}_y \\
\hat{e}_\theta &= -\sin \theta \hat{e}_x + \cos \theta \hat{e}_y \\
\hat{e}_z &= \hat{e}_z
\end{align*} \]
Vectors – Spherical coordinate system

\[ \mathbf{v} = \begin{pmatrix} \mathbf{v}_r \\ \mathbf{v}_\theta \\ \mathbf{v}_\phi \end{pmatrix} = \mathbf{v}_r \hat{e}_r + \mathbf{v}_\theta \hat{e}_\theta + \mathbf{v}_\phi \hat{e}_\phi \]

• The spherical basis vectors are variable (with position)

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

\[ \hat{e}_x = \sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z \]
\[ \hat{e}_y = \cos \theta \cos \phi \hat{e}_x + \cos \theta \sin \phi \hat{e}_y + (-\sin \theta) \hat{e}_z \]
\[ \hat{e}_z = (-\sin \phi) \hat{e}_x + \cos \phi \hat{e}_y \]

1.4: Express forces on surfaces due to fluids

\[ \text{total fluid force on a surface} = \int_S \left[ \hat{n} \cdot \Pi \right] dS \]
\[ \Pi = \text{stress tensor} \]
Example 1.26: In a liquid of density $\rho$, what is the net fluid force on a submerged sphere (a ball or a balloon)? What is the direction of the force and how does the magnitude of the fluid force vary with fluid density?

Solution: We will be able to do this in this course.

From expression for force due to fluid, obtain (spherical coordinates):

$$f = -\rho g R^2 \int_0^{2\pi} \int_0^\pi (H_0 - R \cos \theta) \hat{e}_r \sin \theta d\theta d\phi$$

Already, we can do the math from here.
1.5: Express torques on surfaces due to fluids

\[ \text{total fluid torque} = \iint_S \left( R \times \left( \hat{n} \cdot \Pi \right) \right) dS \]

\[ \Pi = \text{stress tensor} \]
\[ R = \text{lever arm} \]

We shall learn to write the stress tensor for our systems; then we can calculate stresses, torques.

Example 4.2, Torque in Couette Flow: A cup-and-bob apparatus is widely used to measure viscosities for fluids. For the apparatus below, what is the torque needed to turn the inner cylinder (called the bob) at an angular speed of \( \Omega \)?

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Torque in Couette Flow
Solution:

1. Solve for velocity field (microscopic momentum bal)
2. Calculate stress tensor
3. Formulate equation for torque (an integral)
4. Integrate
5. Apply boundary conditions

Summary of Quick Start

A: Mechanical Engineering Balance
   1. SI-SO, steady, incompr, no rxn, no T, no Q
   2. Choose points 1 and 2 wisely
   3. Solve for $F$ or $W_{s, on}$ or $p$, velocity, elevation

B): Use Calculus in Fluid Mechanics to
   1. Calculate flow rate
   2. Calculate average velocity
   3. Express velocity in a fluid as a vector field
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