§ 237. THE FLOW OF HEAT IN A SPHERE AND CONE

The solution useful for small values of \( x/\alpha \) is

\[
v = \frac{1}{2r(\pi \alpha t)^{\frac{3}{2}}} \sum_{n=1}^{\infty} \int_{0}^{\pi} \sin \frac{n \pi r}{a} \left( \frac{a^2}{n \pi a^2} \right)^n \exp \left[ \frac{-(2ma+r'-r)^2}{4\alpha t} \right] - \exp \left[ \frac{-(2ma+r'-r)^2}{4\alpha t} \right] dr'.
\]

XI. Zero surface temperature. Initial temperature

\[
f(r) = b_0 + br + cr^2 + \ldots
\]

For \( r = 0 \), the solution reads

\[
x = \frac{2}{ar^2} \sum_{n=1}^{\infty} \sin \frac{n \pi r}{a} \left( \frac{a^2}{n \pi a^2} \right)^n \left[ \frac{\sin n \pi (n^2 \pi^2 a^4 + 24 - n^4 \pi^4 a^4 + 24 (-1)^n n^4)}{n^4 \pi^4 a^4} \right] \times e^{-\alpha n^2 \pi^2 ta^2},
\]

9.4. The sphere \( 0 < r < \alpha \). Initial temperature \( f(r) \). Radiation at the surface

If the sphere radiates into medium at zero the equations for \( v \) are

\[
\frac{\partial v}{\partial t} = \kappa \left( \frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \frac{\partial v}{\partial r} \right), \quad 0 < r < \alpha,
\]

\[
\frac{\partial v}{\partial r} + hv = 0, \quad \text{when } r = \alpha,
\]

and

\[
v = f(r), \quad \text{when } t = 0.
\]

Putting \( u = vr \), we have

\[
\frac{\partial u}{\partial t} = \kappa \left( \frac{\partial^2 u}{\partial r^2} \right), \quad 0 < r < \alpha,
\]

\[
u = 0, \quad \text{when } r = 0,
\]

and

\[
\frac{\partial u}{\partial r} + \left( \frac{\alpha}{a} - 1 \right) u = 0, \quad \text{when } r = \alpha,
\]

\[
u = rf(r), \quad \text{when } t = 0.
\]

The problem is thus reduced to that of linear flow of heat in a slab, one end being kept at zero temperature, while at the other end radiation takes place into a medium at zero. This problem has been solved in 3.10 (11), and in this we have only to replace \( l \) by \( a \), \( x \) by \( r \), and \( h \) by \( (ah-1)/a \). Thus the solution of (1)-(3) is

\[
v = \frac{2}{ar} \sum_{n=1}^{\infty} e^{-\alpha n^2 \pi^2 ta^2} \frac{a^2}{a^2 + (ah-1)} \sin n \pi r \int_{0}^{\pi} r' f(r') \sin n \pi r' \, dr',
\]

\[\dagger\text{van Onsten, Geophysics, 5 (1940) 75-79. He considers two more powers in the series (22), and gives numerical values for the coefficients of the first forty terms of the series (23).}\]

\[\ddagger\text{This solution is easily obtained directly, cf. C.II., § 65. See also § 14.7 II.}\]
where \( \pm \alpha_n, n = 1, 2, \ldots \) are the roots of
\[
a_n \cot a_n + ah - 1 = 0.
\]

(9)

The equation (9) is simply the equation 3.10 (7) which has already been discussed and whose roots are tabulated in Appendix IV, except that the parameter \( ah \), which was always positive in § 3.11, is replaced by \( ah - 1 \) which may be negative. Provided \( h > 0 \), i.e. \( ah - 1 > -1 \), the remarks of §§ 3.10, 3.11 hold, and the roots of (9) are all real.†

If the initial temperature \( f(r) = V \) constant, (8) becomes\‡

\[
v = \frac{2hV}{r} \sum_{n=1}^{\infty} \frac{e^{-\kappa r}}{a_n^2 + (ah-1)^2} \sin a_n \sin ra_n.
\]

(10)

If the sphere has zero initial temperature and is heated by radiation from medium at temperature \( \kappa t \), the solution is

\[
v = k \left( \frac{r^2 \kappa h}{6 \alpha a} - \alpha^2 (3 + 2a) \right) - \frac{2\alpha \kappa h}{\alpha a} \sum_{n=1}^{\infty} \frac{\sin ra_n}{\alpha_n^2 \alpha_n^2 + ah(ah-1)} \sin \alpha_n e^{-\kappa r},
\]

(11)

where the \( a_n \) are the positive roots of (9).

If the sphere has zero initial temperature and is heated by radiation from medium at temperature \( V \sin(\alpha t + \epsilon) \), the temperature is

\[
v = \frac{2\alpha \kappa h V}{r} \sum_{n=1}^{\infty} \alpha_n^2 (\alpha_n^2 \sin \epsilon - \omega \cos \epsilon) (ah-1) \sin ra_n e^{-\kappa r}.
\]

\[
\alpha_n = \frac{\alpha h A_3}{2aA_2} \sin(\alpha t + \epsilon + \phi_1 - \phi_2),
\]

(12)

where

\[
A_1 = \sinh \omega' r \cos \omega' r + i \cosh \omega' r \sin \omega' r,
\]

\[
A_2 = \alpha \omega' (1 + i) \cosh \omega' (1 + i) + (ah-1) \sinh \alpha \omega' (1 + i),
\]

\[
\omega' = \sqrt{\alpha \omega / 2},
\]

and the \( a_n \) are the positive roots of (9).

9.5. Application to the determination of the conductivities of poor conductors

The expression we have just obtained for the temperature in a sphere cooling by radiation at the surface converges so rapidly that when a sufficient time has passed the terms after the first may be neglected.

† If \( h < 0 \) there is a pair of imaginary roots, but this case is, as always, excluded on physical grounds. If \( h = 0 \), that is, no flow of heat at the surface, (8) has a zero root, and a term

\[
\frac{3}{a^2} \int_0^\beta r^2 f(r) \, dr
\]

has to be added to the value of (8) with \( h = 0 \). Cf. 3.4 (6), 7.3 (3).

‡ Surface and centre temperatures for this case are plotted against \( ah \) for various values of \( a \) by Schack, Stahl u. Eisen, 50 (1930) 1299. See also Heisser, Trans. Amer. Soc. Mech. Engrs. 69 (1947) 227-36.
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