CM3215 Fundamentals of Chemical Engineering Laboratory

Unsteady Heat Transfer to a Sphere—
Measuring the heat transfer coefficient (fitting PDE soln)

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Objective: Measure the heat-transfer coefficient to a 1.0 inch brass sphere dropped into a well-stirred beaker of hot water.

Strategy: ?
Objective: Measure the heat-transfer coefficient to a 1.0 inch brass sphere dropped into a well-stirred beaker of water.

Strategy: ?

To determine our experimental strategy, we begin with: What is heat transfer coefficient?

Newton’s Law of Cooling: \[
\frac{q_r}{A} = h|T_{bulk} - T_{surface}|
\]

Perhaps: measure \(T_{surface}\) and \(T_{bulk}\)?

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How to experimentally measure heat-transfer coefficient?

Newton’s Law of Cooling: \[
\frac{q_r}{A} = h|T_{bulk} - T_{surface}|
\]

Perhaps: measure \(T_{surface}\) and \(T_{bulk}\)?
How to experimentally measure heat-transfer coefficient?

Newton’s Law of Cooling:
(a boundary condition)

\[ \frac{|q_r|}{A} = h |T_{\text{bulk}} - T_{\text{surface}}| \]

Perhaps: measure \( T_{\text{surface}} \) and \( T_{\text{bulk}} \)?

But they will be equal!

Need heat transfer to be taking place.

Unsteady state experiment, then.

It would be awkward to try to measure the surface temperature.
How to experimentally measure heat-transfer coefficient?

Newton’s Law of Cooling: \( \frac{|q_f|}{A} = h(T_{bulk} - T_{surface}) \)

Perhaps: measure \( T_{surface} \) and \( T_{bulk} \)?

But they will be equal!

Need heat transfer to be taking place.

Unsteady state experiment, then.

It would be awkward to try to measure the surface temperature. "We can embed a thermocouple in the center perhaps?"

How is the temperature at the center related to what we seek to measure, \( h \)?

We can model the situation and see what the connection is.
Experiment: Measure T(t) at the center of a sphere:

Initially:
\[ t < t_0 \]
\[ T = T_0 \]

Suddenly:
\[ t \geq t_0 \]
\[ T = T(t) \]

Heat transfer takes place.
**Experiment:** Measure \( T(t) \) at the center of a sphere:

\[
\begin{align*}
\text{Initially:} & \quad t < t_0 \\
& \quad T = T_0 \\
\text{Suddenly:} & \quad t \geq t_0 \\
& \quad T = T(t)
\end{align*}
\]

T-couple measures \( T(t) \) at the center of the sphere.

**Excel:**

<table>
<thead>
<tr>
<th>t(s)</th>
<th>( T(\degree C) )</th>
</tr>
</thead>
<tbody>
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<tr>
<td>2.11E-01</td>
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<td>7.48E+00</td>
</tr>
<tr>
<td>1.37E+00</td>
<td>7.57E+00</td>
</tr>
</tbody>
</table>

**Experiment:** Measure \( T(t) \) at the center of a sphere:

Initially: \( t < t_0 \)  
\[ T = T_0 \]

Suddenly: \( t \geq t_0 \)
[Excel table]

**How do we set up the model for this problem?**

**Where does \( h \) come into it?**

**Modeling exercise:** How does the temperature at the center of a sphere, subjected to this history, change with time?
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**Microscopic Energy Balance**

Microscopic energy balance, constant thermal conductivity; Gibbs notation

\[
\rho C_v \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S
\]

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

\[
\rho C_v \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S
\]

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

\[
\rho C_v \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r^2 \sin \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S
\]

Microscopic energy balance, constant thermal conductivity; spherical coordinates

\[
\rho C_v \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = k \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S
\]

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

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**Modeling exercise**: How does the temperature at the center of a sphere, subjected to this history, change with time?

**Suddenly**:

\[
\begin{align*}
\text{at } t < t_0: & \quad T = T_0 \\
\text{at } t \geq t_0: & \quad T = T(t)
\end{align*}
\]

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

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Unsteady State Heat Transfer to a Sphere

Microscopic energy balance in the sphere:

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( 1 - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \right)
\]

Boundary conditions:
- \( r = R, \quad q_r = -k \frac{\partial T}{\partial r} = h(T(r) - T_{bulk}) \quad t > 0 \)
- \( r = 0, \quad \frac{q_r}{A} = 0 \quad \forall t \)

Initial condition:
- \( t = 0, \quad T = T_{initial} \quad \forall r \)

("\forall" means "for all")

Now, Solve

- \( r = 0, \quad \frac{\partial}{\partial A} = 0 \quad \forall t \)

Initial condition:
- \( t = 0, \quad T = T_{initial} \quad \forall r \)

("\forall" means "for all")
Unsteady State Heat Transfer to a Sphere

Solution: 
\[ \xi(r, t) = \frac{T(r, t) - T_{bulk}}{T_{initial} - T_{bulk}} \]

\[ \text{Bi} = \frac{hR}{k} \quad \text{Fo} = \frac{\alpha t}{R^2} \]

\[ \xi = \frac{T - T_b}{T_i - T_b} = 2Bi \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)} \left( \frac{\sin r\lambda_n}{r\lambda_n} \right) \left( \frac{\sin R\lambda_n}{R\lambda_n} \right) \left( \frac{(R\lambda_n)^2 + (Bi - 1)^2}{(R\lambda_n)^2 + Bi(Bi - 1)} \right) \]

where the eigenvalues \( \lambda_n \) satisfy this equation:

\[ \frac{R\lambda_n}{\tan R\lambda_n} + Bi - 1 = 0 \]

Characteristic Equation

(Carslaw and Yeager, 1959, p237)
Unsteady State Heat Transfer to a Sphere

Solution:

\[ \xi(r, t) \equiv \frac{T(r, t) - T_{\text{bulk}}}{T_{\text{initial}} - T_{\text{bulk}}} \]

Depends on material \((\alpha = k/\rho C_p)\), and heat transfer processes at surface \((h)\)

\[ Bi \equiv \frac{hR}{k}, \quad Fo \equiv \frac{\alpha t}{R^2} \]

\( (Bi = \text{Biot number}; Fo = \text{Fourier number})\)

\[ \xi = \frac{T - T_b}{T_i - T_b} = 2Bi \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left( \frac{\sin r \lambda_n}{r \lambda_n} \right) \left( \frac{\sin R \lambda_n}{R \lambda_n} \right) \left( \frac{(R \lambda_n)^2 + (Bi - 1)^2}{(R \lambda_n)^2} \right) \]

where the eigenvalues \(\lambda_n\) satisfy this equation:

\[ \frac{R \lambda_n}{\tan R \lambda_n} + Bi - 1 = 0 \]

(Carslaw and Yeager, 1959, p237) © Faith A. Morrison, Michigan Tech U.

Unsteady State Heat Transfer to a Sphere

What does this look like?

\[ \xi = \frac{T - T_b}{T_i - T_b} = 2Bi \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left( \frac{\sin r \lambda_n}{r \lambda_n} \right) \left( \frac{\sin R \lambda_n}{R \lambda_n} \right) \left( \frac{(R \lambda_n)^2 + (Bi - 1)^2}{(R \lambda_n)^2} \right) \]

where the eigenvalues \(\lambda_n\) satisfy this equation:

\[ \frac{R \lambda_n}{\tan R \lambda_n} + Bi - 1 = 0 \]

Characteristic Equation

Let’s plot it to find out. (Excel)

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Solution:

\[ \text{Bi} \equiv \frac{hR}{k} \quad \text{Fo} \equiv \frac{at}{R^2} \]

\[ \xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-\text{Fo}(\lambda_n R)^2} \left( \frac{\sin \lambda_n}{r \lambda_n} \right) \left( \frac{\sin \lambda_n R}{R \lambda_n} \right) \left( \frac{(\lambda_n R)^2 + (\text{Bi} - 1)^2}{(\lambda_n R)^2 + \text{Bi}(\text{Bi} - 1)} \right) \]

\[ \xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-\text{Fo}(\lambda_n R)^2} \left( \text{bunch of terms that vary with Bi and } \lambda_n(\text{Bi}) \right) \]

Exponential decay with Fo (scaled time)

If we fix Bi, then only Fo varies:

\[ \lambda_n(\text{Bi}) \text{ varies with Bi: } \frac{R \lambda_n}{\tan R \lambda_n} + \text{Bi} - 1 = 0 \]

Characteristic Equation

Unsteady State Heat Transfer to a Sphere

If we fix Bi, then only Fo varies:

\[ \xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} T_{\text{term}_n} \]

For a fixed Bi:

\[ \text{Bi} \equiv \frac{hR}{k} \]

Sum of 9 terms

Fo \equiv \frac{at}{R^2}
Unsteady State Heat Transfer to a Sphere

If we fix Bi, then only Fo varies:

• It is an infinite sum of terms.
• Each term corresponds to one eigenvalue, \( \lambda_n \).
• The first term \( n = 1 \) is the dominant one.
• The \( n > 1 \) terms alternate in sign (positive and negative).
• Higher terms are “fixing” the short time behavior.

For a fixed Bi

\[
Bi = \frac{hR}{k}
\]

\[
\xi(r, t) \equiv \frac{T(r, t) - T_{bulk}}{T_{in} - T_{bulk}}
\]

\[
Fo \equiv \frac{at}{R^2}
\]

Unsteady State Heat Transfer to a Sphere

If we fix Bi, then only Fo varies:

For \( Fo > 0.2 \), the higher-order terms make no contribution.

\[
\text{result (9 terms)}
\]
Unsteady State Heat Transfer to a Sphere

If we fix Bi, then only Fo varies:

For $Fo > 0.2$, the higher-order terms make no contribution

The higher-order terms are working to get the solution right for shorter and shorter times (low Fo)

We already know the short-time behavior; we thus concentrate on long times to fit the data (and therefore need only one term of the summation)
Unsteady State Heat Transfer to a Sphere

If we fix Bi, then only Fo varies:

\[ \xi(r, t) = \frac{T(r, t) - T_{\text{bulk}}}{T_{\text{initial}} - T_{\text{bulk}}} \]

- Plotted log-linear, the solution is linear for \( Fo > 0.2 \)
- The slope depends on Biot number

\[ \text{Bi} \equiv \frac{hR}{k} \]
\[ \frac{1}{\text{Bi}} = \frac{k}{hR} \]

For a fixed Bi:
- the results are only a function of Fo.

For fixed Bi:
- Solution is an infinite sum of terms.
- Each term corresponds to one eigenvalue, \( \lambda_n \)
- The first term \( n = 1 \) is the dominant one
- The \( n > 1 \) terms alternate in sign (positive and negative)
- Higher terms are "fixing" the short time behavior
- At fixed Biot number, the time-dependence is an exponential decay (for \( Fo > 0.2 \))
- Our Biot number will be fixed; but we not know what it is until we fit our results to the model.

Question: How do various values of Biot number affect the heat transfer that occurs?
Unsteady State Heat Transfer to a Sphere

The roots of the characteristic equation $\lambda_n$ vary with Bi

Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The $\lambda_n$ are the roots of the characteristic equation.
- They depend on Biot number Bi.

$$\text{Bi} \equiv \frac{hR}{k}$$

- At low Bi, the temperature is uniform in the sphere; heat transfer is limited by rate of heat transfer to the surface ($h$).

Low Bi: high $k$, low $h$
Unsteady State Heat Transfer to a Sphere

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the sphere.

High Bi: low k, high h

\[ R\lambda_n \tan R\lambda_n + Bi - 1 = 0 \]

- The \( \lambda_n \) are the roots of the characteristic equation
- They depend on Biot number Bi

\[ Bi \equiv \frac{hR}{k} \]

At moderate Bi, heat transfer is affected by both conduction in the sphere and the rate of heat transfer to the surface.

Moderate Bi: neither process dominates

\[ \tan R\lambda_n + Bi - 1 = 0 \]

- The \( \lambda_n \) are the roots of the characteristic equation
- They depend on Biot number Bi

\[ Bi \equiv \frac{hR}{k} \]
Summary:

- The key to analyzing our data is determining the Biot number for the data.
- We determine the Biot number by fitting the model predictions to our data.
- The value of Biot number that results in the closest fit between model and data tells us our value of $h$.

In the literature, there are handy plots of the solutions to Unsteady State Heat Transfer to a Sphere (and other shapes)

Heisler charts

(Geankoplis; see also Wikipedia)
The Heisler Chart is a catalog of all the $Y_0 = \xi(t)$ long-time shapes for various values of Biot number $\text{Bi} = \frac{hR}{k} = \frac{1}{\text{Pr}}$. 

From Geankoplis, 4th edition, page 374
Our data correspond to only one of these lines; when we know which line, we know Bi.

The Heisler Chart is a catalog of all the \( Y_0 = \xi(t) \) long-time shapes for various values of Biot number \( \text{Bi} = \frac{hr}{k} = \frac{1}{m} \).

All the lines go through the point \( (Fo, Y_0) = (0, 1) \).
Literature solutions to Unsteady State Heat Transfer to a Sphere

Heisler charts (Geankoplis; see also Wikipedia)

Note: the parameter $m$ from the Geankoplis Heisler chart is NOT the slope of the line! It is a label of $\frac{1}{Bi}$.

Also, $x_1$ is the sphere radius

Heisler charts (Geankoplis; see also Wikipedia)

Note also: think of it as 4 separate graphs
Assignment 8: Unsteady Heat Transfer to a Sphere

(team assignment)

• Using the time-dependent temperature versus time data measured by colleagues (supplied to you), calculate the heat transfer coefficient for the laboratory experiment (sphere in a beaker).

• Report the heat transfer coefficient, \( h \).

• Report the Biot number.

• Indicate which physics is dominating the dynamics: heat transfer to the sphere (\( h \)), heat transfer within the sphere (\( k \)), or neither dominate.

• Describe your method for obtaining Bi and \( h \) (briefly; you may give steps as a list in your memo; it’s not a report).

• Conduct and report an uncertainty analysis to determine the error limits on \( h \).
Pre-lab Assignment (due Tuesday in lab):

• In your lab notebook, outline how you plan to proceed; show Sajjad.
• Obtain $\alpha = k / \rho C_p$ and $k$ for brass ahead of time and record both in notebook (with reference)
• Have your pump characteristic curve in your lab notebook for collection by Sajjad (for the archive)

Assignment 8: Unsteady Heat Transfer to a Sphere

(team assignment)

• Using the time-dependent temperature versus time data measured by colleagues (supplied to you), calculate the heat transfer coefficient for the laboratory experiment (sphere in a beaker).
• Report the heat transfer coefficient, $h$.
• Report the Biot number.
• Indicate which physics is dominating the dynamics: heat transfer to the sphere ($h$), heat transfer within the sphere ($k$), or neither dominate.
• Describe your method for obtaining $\text{Bi}$ and $h$ (briefly; you may give steps as a list in your memo; it’s not a report).
• Conduct and report an uncertainty analysis to determine the error limits on $h$. 
Detection Limit (recall from Rotameter Lab)

What’s the **lowest** accurate value of a quantity?

\[
\Delta p = \Delta p_{\text{predicted}} \pm 2s_{y_p}
\]

\[
(\Delta p; \text{psi}) = (\bar{m})(I; mA) + b
\]

\[2s_{y_p} = y_p \Rightarrow 100\% \text{ error}\]
Detection Limit (recall from Rotameter Lab)

What's the **lowest** accurate value of a quantity?

\[ \Delta p = \Delta p_{\text{predicted}} \pm 2s_{y_p} \]

- \(4s_{y_p} = y_p\) \(\Rightarrow\) 50% error
- \(2s_{y_p} = y_p\) \(\Rightarrow\) 100% error

Detection Limit (recall from Rotameter Lab)

What’s the **lowest** accurate value of a quantity?

\[ \Delta p = \Delta p_{\text{predicted}} \pm 2s_{y_p} \]

- \(8s_{y_p} = y_p\) \(\Rightarrow\) 25% error
- \(4s_{y_p} = y_p\) \(\Rightarrow\) 50% error
- \(2s_{y_p} = y_p\) \(\Rightarrow\) 100% error
**Detection Limit (recall from Rotameter Lab)**

What's the **lowest** accurate value of a quantity?

\[
\Delta p = \Delta p_{predicted} \pm 2\sigma_p
\]

The choice is up to you.

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<table>
<thead>
<tr>
<th>( f(x_1, x_2, x_3, x_4, x_5) ):</th>
<th>Formula for ( f ):</th>
<th>Representative value of ( f ): (include units)</th>
<th>95% C.I. of ( f ): (( f \pm 2\sigma_f )) (include units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y_0 = \frac{T_m - T}{T_m - T_0} )</td>
<td>( T_m )</td>
<td>unitless</td>
<td>( 1.56 \times 10^{-2} )</td>
</tr>
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<td>( 0.55^\circ C )</td>
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<td></td>
<td>( 99^\circ C )</td>
<td>( 0.55^\circ C )</td>
</tr>
</tbody>
</table>

\[
\Delta p = \Delta p_{predicted} \pm 2\sigma_p
\]

Detection limit (25% error): \( 8\sigma_{sf} = 6.25 \times 10^{-2} \)