Exam I
CM4650
Polymer Rheology
10 February 2015

Please be neat.

This exam is closed book, closed notes. No internet-capable devices are permitted. Submit only your own work.

Navier-Stokes Equation (Gibbs notation): \[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

Continuity Equation (Gibbs notation): \[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \]

Newtonian Incompressible Constitutive Equation (Gibbs notation): \[ \varepsilon = -\mu \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right] \]

Fluid tension force on a surface S: \[ \mathbf{F}_{on} = \iint_S \left[ \hat{n} \cdot \left( -\nabla \right) \right]_{surface} dS \]

Flow rate through surface S: \[ Q = \iint_S \left[ \hat{n} \cdot \mathbf{v} \right]_{surface} dS \]

Fluid torque on a surface S: \[ \mathbf{T}_{on} = \iint_S \left[ \mathbf{R} \times \left( \hat{n} \cdot \left( -\nabla \right) \right) \right]_{surface} dS \]

(R is the lever-arm vector from the axis of rotation to the location at which force is applied)

| Coordinate system | coordinates \( x = r \sin \theta \cos \phi \) \( y = r \sin \theta \sin \phi \) \( z = r \cos \theta \) \( x = r \cos \theta \) \( y = r \sin \theta \) \( z = z \) | basis vectors \( \hat{e}_r = (\sin \theta \cos \phi)\hat{e}_x + (\sin \theta \sin \phi)\hat{e}_y + \cos \theta \hat{e}_z \) \( \hat{e}_\theta = (\cos \theta \cos \phi)\hat{e}_x + (\cos \theta \sin \phi)\hat{e}_y + (-\sin \theta)\hat{e}_z \) \( \hat{e}_\phi = (-\sin \phi)\hat{e}_x + (\cos \phi)\hat{e}_y \) \( \hat{e}_r = (\cos \theta)\hat{e}_x + (\sin \theta)\hat{e}_y \) \( \hat{e}_\theta = (-\sin \theta)\hat{e}_x + (\cos \theta)\hat{e}_y \) \( \hat{e}_z = \hat{e}_z \) |
Cylindrical Coordinate System: Note that the $\theta$-coordinate swings around the $z$-axis and the $r$-coordinate is perpendicular to the $z$-axis.

![Cylindrical Coordinate System Diagram](image)

Spherical Coordinate System: Note that the $\theta$-coordinate swings down from the $z$-axis and the $r$-coordinate emits radially from the origin; these are different from their definitions in the cylindrical system above.

![Spherical Coordinate System Diagram](image)
1. (20 points) Use Einstein notation to write this expression in terms of Cartesian components: \((a \cdot \nabla u)\). Note that \(a\) and \(u\) are both variables. Expand any derivatives of products in your final answer. Insert the appropriate summation signs in your final answer.

2. (20 points) Consider steady laminar flow inside a horizontal pipe (circular cross-section, radius \(R\), length \(L\)). The flow problem is solved in cylindrical coordinates with flow in the \(z\)-direction. Gravity is in the negative \(y\)-direction of the Cartesian coordinate system (The geometries of the two coordinate systems are shown on page 2 of the exam). What are the components of gravity written in the cylindrical coordinate system?

\[
g = \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}_{xyz} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}_{r\theta z}
\]

3. (20 points) Consider steady laminar flow inside a horizontal pipe (circular cross-section, radius \(R\), length \(L\)). The flow is solved in cylindrical coordinates with flow in the \(z\)-direction. The stress tensor \(\Pi\) for this flow is given below. What is the force on the walls of the pipe in terms of the quantities given below? I’m looking for an answer expressed as an integral over vector components involving the velocity and pressure functions. Please set up the integral with the information given (you cannot complete the integration); include the limits of integration. The general expression for fluid force on a surface is given below (and it is also on page 1 of the exam).

\[
E_{on} = \int \int_{S} [\hat{n} \cdot (-\Pi)]_{surface} dS
\]

\[
\Pi = pI - \mu(\nabla v + (\nabla v)^T) = \begin{pmatrix} p(z) & 0 & -\mu \frac{\partial v_z}{\partial r} \\ 0 & p(z) & 0 \\ -\mu \frac{\partial v_z}{\partial r} & 0 & p(z) \end{pmatrix}_{r\theta z}
\]
4. (20 points) What are the boundary conditions for the flow sketched below? An incompressible, Newtonian fluid is confined in the gap between two cylinders. The inner cylinder, with radius $\kappa R$, turns at a steady angular velocity $\Omega$. The outer cylinder, of radius $R$, is stationary. The cylinder is very long; the flow is steady.

$$\mathbf{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}_{r\theta z}$$

**cross-section A:**

- $\kappa R$
- $R$
- $\theta$
5. (20 points) An incompressible, Newtonian fluid is placed between two very long (length = \( L \)), very wide (width = \( W \)), parallel plates separated by a gap \( B \). The top plate is pulled to the right at a steady speed of \( V/2 \) and the bottom plate is pulled to the left at a steady speed of \( V/2 \). There is no imposed pressure difference, and the entire flow is open to atmospheric pressure. Solve for the steady state velocity field in the gap. Use the coordinate system shown in the figure below, which is centered between the two plates.