FINAL EXAM
CM4650
28 Ap 2009
SOLN

1. (a) Example of mat'l function:

shear viscosity $\eta(\dot{\gamma})$
shear normal stress coeff Ψ_1, Ψ_2
elongational viscosity $\bar{\eta}(t)$
elastic modulus $G'(\omega)$
etc.

(b) Example of constitutive eqn

$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}}$ Newtonian

$\underline{\underline{\tau}} = -\eta \dot{\underline{\underline{\gamma}}}$ generalized Newtonian

$\underline{\underline{\tau}} = -\int_{-\infty}^{t} G(t-t') \dot{\underline{\underline{\gamma}}}(t')\, dt'$

GLVE etc.

c) capable of predicting memory:

Maxwell

GLVE

Lodge

d) TRUE!

2) $\nabla(\underline{\underline{A}} \cdot \underline{u}) = ?$

$$A_{ij}\, \underbrace{\hat{e}_i \hat{e}_j \cdot u_p \hat{e}_p}_{\delta_{jp}}$$

"j becomes p"

$$= A_{ip} \hat{e}_i u_p = A_{ip} u_p \hat{e}_i$$

$$\nabla(\underline{\underline{A}} \cdot \underline{u}) = \frac{\partial}{\partial x_s} \hat{e}_s \; A_{ip} u_p \hat{e}_i$$

$$= \frac{\partial}{\partial x_s}(A_{ip} u_p)\, \hat{e}_s \hat{e}_i$$

$$\nabla\left(\underline{\underline{A}}\cdot\underline{u}\right) = \left(A_{ip}\frac{\partial u_p}{\partial x_s} + u_p\frac{\partial A_{ip}}{\partial x_s}\right)\hat{e}_s\,\hat{e}_i$$

21-component: $s=2$
$i=3$

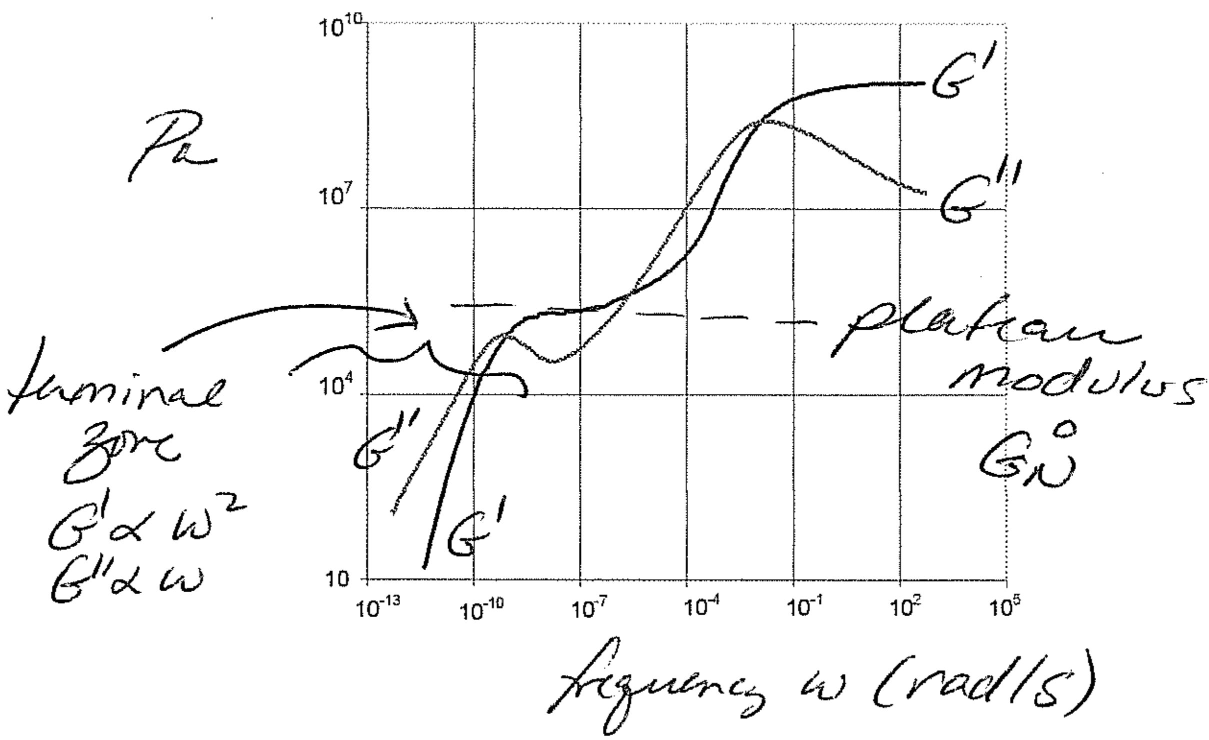
$$\boxed{\left[\nabla\left(\underline{\underline{A}}\cdot\underline{u}\right)\right]_{21} = \sum_{p=1}^{3}\left(A_{3p}\frac{\partial u_p}{\partial x_2} + u_p\frac{\partial A_{3p}}{\partial x_2}\right)}$$

4

Soln #3

3. (10 points) The small-amplitude oscillatory shear functions G' and G" for a polymer melt are plotted below. Place the following labels on the graph:

   a) Label one curve G' and the other curve G"
   b) Label the x-axis (what is the variable, and what would be reasonable units?)
   c) Label the y-axis with reasonable units
   d) Indicate the plateau modulus
   e) Indicate which region is the "terminal zone"



4. (25 points)

   a) Calculate steady shear viscosity η for the constitutive equation given below; g, λ, and ϕ_1 are constant parameters of the model. Please express your final answer as a simple integral over $s = t-t'$.

   b) Does this model predict a nonzero Ψ_2 in steady shear flow? Justify your answer, but you do not need to calculate Ψ_2.

   c) **BONUS** (5 points): carry out the integral from part a) and obtain the final prediction for the steady shear viscosity η.

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \frac{g}{\lambda} e^{-\frac{t-t'}{\lambda}} \left[\phi_1 \underline{\underline{C}}^{-1}(t',t) + \underline{\underline{C}}(t,t') \right] dt'$$

4.

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{g}{\lambda} e^{\frac{-(t-t')}{\lambda}} \left(\phi_1 \underline{\underline{C}}^{-1} + \underline{\underline{C}} \right) dt'$$

Steady shear: TABLE 9.3

$$\underline{\underline{C}} = \begin{pmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{C}}^{-1} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$$

$$\eta = \frac{-\tau_{21}}{\dot{\gamma}_0} = +\frac{1}{\dot{\gamma}_0} \left[\int_{-\infty}^{t} \frac{g}{\lambda} e^{\frac{-(t-t')}{\lambda}} \left(\phi_1 \gamma - \gamma \right) dt' \right]$$

where $\gamma = \int_{t'}^{t} \dot{\gamma}_0 \, dt''$

$= \dot{\gamma}_0 (t - t')$

$$\eta = \frac{g}{\dot{\gamma}_0 \lambda} \int_{-\infty}^{t} e^{-\frac{(t-t')}{\lambda}} \underset{\dot{\gamma}_0 (t-t')}{\underset{\uparrow}{\gamma}} (\phi_1 - 1)\, dt'$$

$$\eta = \frac{g(\phi_1 - 1)}{\lambda} \int_{-\infty}^{t} e^{-\frac{(t-t')}{\lambda}} (t-t')\, dt'$$

let $s = t - t'$

$ds = -dt'$

$t' = -\infty \quad s = \infty$

$t' = t \quad s = 0$

$$\eta = \frac{-g(\phi_1 - 1)}{\lambda} \int_{\infty}^{0} e^{-s/\lambda} s\, ds$$

$$\boxed{\eta = \frac{g(\phi_1 - 1)}{\lambda} \int_{0}^{\infty} e^{-s/\lambda} s\, ds}$$

BONUS

$$\int_0^\infty e^{-s/\lambda} s \, ds$$

$$u = s \qquad dv = e^{-s/\lambda} ds$$

$$du = ds \qquad v = -\lambda e^{-s/\lambda}$$

$$= -s\lambda e^{-s/\lambda} \Big|_0^\infty + -\lambda \int_0^\infty \lambda e^{-s/\lambda} \left(\frac{ds}{-\lambda}\right)$$

$$= -\lambda^2 e^{-s/\lambda} \Big|_0^\infty = -\lambda^2 (0 - 1)$$

$$= \lambda^2$$

$$\therefore \eta = g \frac{(\phi_1 - 1)}{\cancel{\lambda}} \lambda^{\cancel{2}}$$

$$\boxed{\eta = g\lambda(\phi_1 - 1)}$$

5 PL GNF wire coating

<u>MASS</u> $\nabla \cdot \underline{v} = 0 = \boxed{\frac{\partial v_z}{\partial z} = 0}$ $\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{r\theta z}$

<u>momentum</u>

$$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{(\underline{v}\cdot\nabla\underline{v})}}_{\text{unidir}}\right) = \underbrace{\cancel{-\nabla p}}_{\text{const}} - \nabla\cdot\underline{\underline{\tau}} + \underbrace{\cancel{\rho \underline{g}}}_{0}$$

$$\boxed{\nabla\cdot\underline{\underline{\tau}} = 0}$$

$$\underline{\underline{\tau}} = -\eta\underline{\underline{\dot\gamma}} \qquad \eta = m\dot\gamma^{n-1}$$

$$\underline{\underline{\dot\gamma}} = \nabla\underline{v} + (\nabla\underline{v})^T$$

(9)

$$\nabla \underline{v} = \begin{pmatrix} \cancel{\frac{\partial v_r}{\partial r}} & \cancel{\frac{\partial v_\theta}{\partial r}} & \frac{\partial v_z}{\partial r} \\ \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta}{r}} & \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r}{r}} & \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta}} \\ \cancel{\frac{\partial v_r}{\partial z}} & \cancel{\frac{\partial v_\theta}{\partial z}} & \cancel{\frac{\partial v_z}{\partial z}} \end{pmatrix}_{r\theta z}$$

θ sym

$\nabla \cdot \underline{v} = 0$

$$\nabla \underline{v} = \begin{pmatrix} 0 & 0 & \frac{\partial v_z}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$\dot{\underline{\underline{\gamma}}} = \begin{pmatrix} 0 & 0 & \frac{\partial v_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial v_z}{\partial r} & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$|\dot{\underline{\underline{\gamma}}}| = \sqrt{\frac{2\left(\frac{\partial v_z}{\partial r}\right)^2}{2}} = \pm\frac{\partial v_z}{\partial r} = \boxed{-\frac{\partial v_z}{\partial r} > 0}$$

FROM TABLE:

no z-dep of v_z

$$\nabla \cdot \underline{\underline{\tau}} = \begin{pmatrix} 0 + 0 + \cancel{\frac{\partial \tau_{zr}}{\partial z}} - 0 \\ 0 + 0 + 0 + 0 \\ \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) + 0 + 0 \end{pmatrix}_{r\theta z}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) \end{pmatrix}_{r\theta z}$$

EOM

$$\nabla \cdot \underline{\underline{\tau}} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) = 0$$

$$\frac{d}{dr}\underbrace{(r\,\tau_{rz})}_{Y} = 0$$

$$\frac{dY}{dr} = 0 \implies \boxed{Y = c_1}$$

$$\Upsilon = r\,\tau_{rz} = c_1$$

$$\boxed{\tau_{rz} = \frac{c_1}{r}}$$

$$\underline{\underline{\tau}} = -\eta\,\dot{\underline{\underline{\gamma}}}$$

$m\dot{\gamma}^{n-1}$

$$\tau_{rz} = -\eta\,\dot{\gamma}_{rz} = -\eta\,\frac{\partial v_z}{\partial r}$$

$$\tau_{rz} = -m\left(-\frac{\partial v_z}{\partial r}\right)^{n-1}\left(\frac{\partial v_z}{\partial r}\right)$$

$$\tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^{n} = \frac{c_1}{r}$$

$$\left(-\frac{dv_z}{dr}\right)^{n} = \frac{c_1}{m}\,\frac{1}{r}$$

$$-\frac{dv_z}{dr} = \left(\frac{c_1}{m}\right)^{\frac{1}{n}}\frac{1}{r^{\frac{1}{n}}}$$

$$\frac{dv_z}{dr} = -\left(\frac{c_1}{m}\right)^{\frac{1}{n}} r^{-\frac{1}{n}}$$

integrating,

$$\boxed{V_z = -\left(\frac{C_1}{m}\right)^{\frac{1}{n}} \frac{r^{-\frac{1}{n}+1}}{\left(-\frac{1}{n}+1\right)} + C_2}$$

BC: $r = \kappa R \quad V_z = V$

$r = R \quad V_z = 0$