Rate of deformation tensor: \( \gamma = \nabla \mathbf{v} + (\nabla \mathbf{v})^T \)

Rate of deformation: \( \dot{\gamma} = \left| \dot{\gamma} \right| \)

Tensor magnitude: \( A = \left| A \right| = +\sqrt{\frac{4A^4}{2}} \)

Shear strain: \( \gamma_{21}(t_a, t_b) = \int_{t_a}^{t_b} \dot{\gamma}_{21}(t'') \, dt'' \)

Navier-Stokes Equation:
\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

Cauchy Momentum Equation:
\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

Continuity Equation:
\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})
\]

Fluid force \( \mathbf{F} \) on a surface \( S \):
\[
\mathbf{F} = \iint_S [\hat{n} \cdot (\mathbf{F} - \mathbf{P})]_{surface} \, dA
\]

Flow rate \( \mathbf{Q} \) through a surface \( S \):
\[
\mathbf{Q} = \iint_S [\hat{n} \cdot \mathbf{v}]_{surface} \, dA
\]

Fluid torque \( \mathbf{T} \) on a surface \( S \): (\( \mathbf{R} \) is the vector from the axis of rotation to the point of application of the force)
\[
\mathbf{T} = \iint_S [\mathbf{R} \times (\hat{n} \cdot \mathbf{F})]_{surface} \, dA
\]
Newtonian, incompressible fluid: \( \tau = -\mu (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \)

Generalized Newtonian fluid (GNF): \( \tau = -\eta (\dot{\gamma}) \dot{\gamma} \)

Power-law GNF model: \( \eta (\dot{\gamma}) = m \dot{\gamma}^{n-1} \)

(Note that \( m \) and \( n \) are parameters of the model and are constants)

Carreau-Yasuda GNF model: \( \eta (\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty) [1 + (\dot{\gamma} \lambda)^a]^{\frac{n-1}{a}} \)

(Note that \( a \), \( \lambda \) and \( n \), \( \eta_0 \) and \( \eta_\infty \) are parameters of the model and are constants)

Elongational flow (uniaxial, biaxial): \( \mathbf{v} = \left( \frac{-\dot{\gamma}(t)x_1}{2} \right) \)

Shear flow: \( \mathbf{v} = \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix} \)

Steady shearing kinematics: \( \dot{\gamma}(t) = \dot{\gamma}_0 \) for all values of time \( t \)

Start-up of steady shearing kinematics: \( \dot{\gamma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases} \)

Cessation of steady shearing kinematics: \( \dot{\gamma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases} \)

Steady elongational kinematics: \( \dot{\varepsilon}(t) = \dot{\varepsilon}_0 \)

Start-up of steady elongation kinematics: \( \dot{\varepsilon}(t) = \begin{cases} 0 & t < 0 \\ \dot{\varepsilon}_0 & t \geq 0 \end{cases} \)

Cessation of steady elongation kinematics: \( \dot{\varepsilon}(t) = \begin{cases} \dot{\varepsilon}_0 & t < 0 \\ 0 & t \geq 0 \end{cases} \)

Shear viscosity: \( \eta = \frac{-(r_{21})}{\dot{\gamma}_0} \)

Shear normal stress coefficients: \( \Psi_1 = \frac{-(r_{11} - r_{22})}{\dot{\gamma}_0^2}, \Psi_2 = \frac{-(r_{22} - r_{33})}{\dot{\gamma}_0} \)

Elongational viscosity: \( \overline{\eta} = \frac{-(r_{33} - r_{11})}{\dot{\varepsilon}_0} \)
**Cylindrical Coordinate System:** Note that the $\theta$-coordinate swings around the $z$-axis and the $r$-coordinate is perpendicular to the $z$-axis.

**Spherical Coordinate System:** Note that the $\theta$-coordinate swings down from the $z$-axis and the $r$-coordinate emits radially from the origin; these are different from their definitions in the cylindrical system above.

<table>
<thead>
<tr>
<th>System</th>
<th>Coordinates</th>
<th>Basis vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical</td>
<td>$x = r \sin \theta \cos \phi$</td>
<td>$\hat{e}_r = (\sin \theta \cos \phi)\hat{e}_x + (\sin \theta \sin \phi)\hat{e}_y + \cos \theta \hat{e}_z$</td>
</tr>
<tr>
<td>Spherical</td>
<td>$y = r \sin \theta \sin \phi$</td>
<td>$\hat{e}_\theta = (\cos \theta \cos \phi)\hat{e}_x + (\cos \theta \sin \phi)\hat{e}_y + (- \sin \theta)\hat{e}_z$</td>
</tr>
<tr>
<td>Spherical</td>
<td>$z = r \cos \theta$</td>
<td>$\hat{e}_\phi = (- \sin \phi)\hat{e}_x + \cos \phi \hat{e}_y$</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>$x = r \cos \theta$</td>
<td>$\hat{e}_r = \cos \theta \hat{e}_x + \sin \theta \hat{e}_y$</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>$y = r \sin \theta$</td>
<td>$\hat{e}_\theta = (- \sin \theta)\hat{e}_x + \cos \theta \hat{e}_y$</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>$z = z$</td>
<td>$\hat{e}_z = \hat{e}_z$</td>
</tr>
</tbody>
</table>
Table of Integrals

\[ \int s^n \, du = \frac{s^{n+1}}{n+1} + C \quad n \text{ is a constant} \]

\[ \int e^s \, ds = e^s + C \]
\[ \int se^s \, ds = e^s(s - 1) + C \]
\[ \int s^2e^s \, ds = e^s(s^2 - 2s + 2) + C \]

\[ \int \cos(s) \, ds = \sin(s) + C \]
\[ \int \sin(s) \, ds = -\cos(s) + C \]
\[ \int s \cos(s) \, ds = s \sin(s) + \cos(s) + C \]
\[ \int s \sin(s) \, ds = \sin(s) - s \cos(s) + C \]
Tensor Invariants

\[ I_B \equiv \sum_{i=1}^{3} B_{ii} = \text{trace}(B) \]

\[ II_B \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij}B_{ji} = \text{trace}(B \cdot B) \]

\[ III_B \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} B_{ij}B_{jk}B_{ki} = \text{trace}(B \cdot B \cdot B) \]

Note that \( |B| = +\sqrt{\frac{II_B}{2}} \)
Table 9.3: Strain tensors for shear and extension in Cartesian coordinates.

For shear flows \( \gamma = \gamma(t', t) = \int_{t'}^{t} \xi(t'') \, dt'' = \int_{t'}^{t} \gamma_{21}(t'') \, dt'' \) and for elongational flows \( \epsilon = \epsilon(t', t) = \int_{t'}^{t} \dot{\varepsilon}(t'') \, dt'' \). The angle \( \psi \) is the angle from \( \Sigma(t) = \mathbf{r} \) to \( \Sigma(t') = \mathbf{r}' \) in counterclockwise (ccw) rotation around the \( \mathbf{\hat{e}}_3 \)-axis.

Errata: www.chem.mtu.edu/~fmorriso/URerrata.html