Chapter 8: Memory Effects: GLVE

Maxwell’s model combines viscous and elastic responses in series:

Spring (elastic) and dashpot (viscous) in series:

Displacements are additive:

\[ D_{\text{total}} = D_{\text{spring}} + D_{\text{dashpot}} \]

Fluids with Memory - Chapter 8

We seek a constitutive equation that includes memory effects.

\[ \tau(t) = f(\dot{\gamma}, I_2, II_2, III_2, \text{material information}) \]

2 equations so far:

\[ \tau(t) = -\mu \dot{\gamma}(t) \]

\[ \tau(t) = -\eta(\dot{\gamma})\ddot{\gamma}(t) \]

So far, stress at \( t \) depends on rate-of-deformation \textbf{at} \( t \) only.
### Current Constitutive Equations

**Newtonian**

\[ \tau(t) = -\mu \dot{\gamma}(t) \]

**Generalized Newtonian**

\[ \tau(t) = -\eta(\dot{\gamma})\dot{\gamma}(t) \]

\[ \dot{\gamma} = \left| \dot{\gamma} \right| \]

**Neither can predict:**

- Shear normal stresses - *this will be wrong so long as we use constitutive equations proportional to \( \dot{\gamma} \)
- Stress transients in shear (startup, cessation) - *this flaw seems to be related to omitting fluid memory*

*We will try to fix this now; we will address the first point when we discuss advanced constitutive equations*

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**Startup of Steady Shearing**

\[ \mathbf{v} = \begin{cases} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{cases}, \quad \dot{\mathbf{v}}(t) = \begin{cases} 0 & t < 0 \\ \dot{\mathbf{v}}_0 & t \geq 0 \end{cases} \]

\[ \eta^+ = \frac{-\tau_{ij}(t)}{\dot{\gamma}_0} \]

*Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB soln*
How can we incorporate time-dependent effects?

First we explore a simple memory fluid.

Let’s construct a new constitutive equation that remembers the stress at a time $t_0$ seconds ago

$$\tau(t) = \tilde{\eta} \dot{\gamma}(t) + (0.8 \tilde{\eta}) \dot{\gamma}(t - t_0)$$

This is the rate-of-deformation tensor $t_0$ seconds before time $t$

Newtonian contribution

$\tilde{\eta}$ is a constant parameter of the model
What does this model predict?

**Steady shear**

\[ \eta = ? \]

\[ \Psi_1 = ? \]

\[ \Psi_2 = ? \]

**Shear start-up**

\[ \eta^+(t) = ? \]

\[ \Psi_1^+(t) = ? \]

\[ \Psi_2^+(t) = ? \]

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**Steady Shear Flow Material Functions**

**Kinematics:**

\[
\mathbf{y} = \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123}
\]

\[ \dot{\gamma}(t) = \dot{\gamma}_0 = \text{constant} \]

**Material Functions:**

\[ \eta = -\frac{\tau_{21}}{\dot{\gamma}_0} \]

First normal-stress coefficient

\[ \Psi_1 \equiv -\frac{(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} \]

Second normal-stress coefficient

\[ \Psi_2 \equiv -\frac{(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} \]
### Start-up of Steady Shear Flow Material Functions

**Kinematics:**

\[
\dot{\chi}(t) = \begin{cases} 
  \dot{\chi}(t)_{x_2} & 0 < t < 0 \\
  0 & t \geq 0 
\end{cases}
\]

\[
\dot{\gamma}(t) = \begin{cases} 
  0 & t < 0 \\
  \dot{\gamma}_0 & t \geq 0 
\end{cases}
\]

**Material Functions:**

- First normal-stress growth function
  \[
  \psi_1^+ = -\frac{\tau_{21}(t)}{\dot{\gamma}_0}
  \]

- Second normal-stress growth function
  \[
  \psi_2^+ = -\frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0^2}
  \]

- Shear stress growth function
  \[
  \eta^+ = \frac{-\tau_{21}(t)}{\dot{\gamma}_0}
  \]

### Cessation of Steady Shear Flow Material Functions

**Kinematics:**

\[
\dot{\chi}(t) = \begin{cases} 
  \dot{\chi}(t)_{x_2} & 0 < t < 0 \\
  0 & t \geq 0 
\end{cases}
\]

\[
\dot{\gamma}(t) = \begin{cases} 
  \dot{\gamma}_0 & t < 0 \\
  0 & t \geq 0 
\end{cases}
\]

**Material Functions:**

- First normal-stress decay function
  \[
  \psi_1^- = -\frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0^2}
  \]

- Second normal-stress decay function
  \[
  \psi_2^- = -\frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0^2}
  \]

- Shear stress decay function
  \[
  \eta^- = \frac{-\tau_{21}(t)}{\dot{\gamma}_0}
  \]
Predictions of the simple memory fluid

\[-\tau(t) = \tilde{\eta} \dot{\gamma}(t) + (0.8\tilde{\eta}) \dot{\gamma}(t - t_0)\]

**Steady shear**

\[\eta = 1.8\tilde{\eta} \]

\[\Psi_1 = \Psi_2 = 0\]

The steady viscosity reflects contributions from what is currently happening and contributions from what happened \(t_0\) seconds ago.

**Shear start-up**

\[\eta^+(t) = \begin{cases} 0 & t < 0 \\ \eta & 0 \leq t \leq t_0 \\ 1.8\tilde{\eta} & t \geq t_0 \end{cases} \]

\[\Psi_1^+(t) = \Psi_2^+(t) = 0\]

Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB solns
Predictions of the simple memory fluid

What the data show:

Adding that contribution from the past introduces the observed “build-up” effect to the predicted start-up material functions.

What the simple memory fluid model predict:

What the GNF models predict:

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We can make the stress rise smoother by adding more fading memory terms.

\[-\tau(t) = \tilde{\eta} \dot{\gamma}(t) + (0.8\tilde{\eta})\dot{\gamma}(t-t_0) + (0.6\tilde{\eta})\dot{\gamma}(t-2t_0)\]

The memory is fading

Newtonian contribution

contribution from \(t_0\) seconds ago

contribution from \(2t_0\) seconds ago

The fit can be made to be perfectly smooth by using a sum of exponentially decaying terms as the weighting functions.

\[-\tau(t) = \tilde{\eta} \left[ \dot{\gamma}(t) + (0.37)\dot{\gamma}(t-t_0) + (0.14)\dot{\gamma}(t-2t_0) + (0.05)\dot{\gamma}(t-3t_0) + \ldots \right] \]

\[= -\tilde{\eta} \sum_{\rho=0}^{\infty} e^{-\rho t_0/\lambda} \dot{\gamma}(t - \rho t_0)\]

\(\eta^+(t)\)

\(t\)

\(t_0, \lambda\) scales the decay

<table>
<thead>
<tr>
<th>(\rho t_0/\lambda)</th>
<th>(e^{-\rho t_0/\lambda})</th>
</tr>
</thead>
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<tr>
<td>0</td>
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<td>3</td>
<td>0.05</td>
</tr>
<tr>
<td>4</td>
<td>0.02</td>
</tr>
</tbody>
</table>
New model:
\[ \tau(t) = \tilde{\eta} \sum_{p=0}^{\infty} e^{-pt_0/\lambda} \dot{\gamma}(t - pt_0) \]

This sum can be rewritten as an integral.
\[ I = \int_{a}^{b} f(x)dx \equiv \lim_{N \to \infty} \left[ \sum_{i=1}^{N} f(a + i\Delta x)\Delta x \right], \quad \Delta x = \frac{b-a}{N} \]

\( a \to t \)
\( x \to -t' \)
\( \Delta x \to -\Delta t' \)
\( i\Delta x - pt_0 = -p\Delta t' \)
\( f(a + i\Delta x) \to e^{-p\Delta t} \dot{\gamma}(t - p\Delta t') \)

(Actually, it takes a bit of renormalizing to make this transformation actually work.)

With proper reformulation, we obtain:

Maxwell Model (integral version)
\[ \tau(t) = -\int_{-\infty}^{t} \left( \frac{\eta_0}{\lambda} \right) e^{-(t-t')/\lambda} \dot{\gamma}(t') dt' \]

Two parameters:
- Zero-shear viscosity \( \eta_0 \) – gives the value of the steady shear viscosity
- Relaxation time \( \lambda \) - quantifies how fast memory fades

Steps to here:
- Add information about past deformations
- Make memory fade
We’ve seen that including terms that invoke past deformations (fluid memory) can improve the constitutive predictions we make.

This same class of models can be derived in differential form, beginning with the idea of combining viscous and elastic effects.

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**The Maxwell Models**

The basic Maxwell model is based on the observation that at long times viscoelastic materials behave like Newtonian liquids, while at short times they behave like elastic solids.

*Hooke’s Law for elastic solids*  \[ \tau_{21} = -G \gamma_{21} \]

*Newton’s Law for viscous liquids*  \[ \tau_{21} = -\eta \dot{\gamma}_{21} \]
Maxwell’s model combines viscous and elastic responses in series

Spring (elastic) and dashpot (viscous) in series:

\[ D_{total} = D_{spring} + D_{dashpot} \]

Displacements are additive:

\[ f = - \frac{dD_{spring}}{dt} \]

\[ f = - \mu \frac{dD_{dash}}{dt} \]

\[ D_{total} = D_{spring} + D_{dash} \]

\[ \frac{dD_{total}}{dt} = \frac{dD_{spring}}{dt} + \frac{dD_{dash}}{dt} \]

\[ f + \frac{\mu}{G_{sp}} \frac{df}{dt} = -\mu \frac{dD_{total}}{dt} \]

© Faith A. Morrison, Michigan Tech U.
\[ f + \frac{\mu}{G_{sp}} \frac{df}{dt} = -\mu \frac{dD_{total}}{dt} \]

By analogy:

\[ \tau_{21} + \frac{\eta_0}{G} \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21} \quad \text{shear} \]

\[ \tau + \frac{\eta_0}{G} \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} \quad \text{all flows} \]

Two parameter model:

\[ \lambda = \frac{\eta_0}{G} \quad \text{Relaxation time} \]

\[ \eta_0 \quad \text{Viscosity} \]

---

The Maxwell Model

\[ \tau + \frac{\eta_0}{G} \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} \]

Two parameter model:

\[ \lambda = \frac{\eta_0}{G} \quad \text{Relaxation time} \]

\[ \eta_0 \quad \text{Viscosity} \]
How does the Maxwell model behave at steady state? For short time deformations?

\[ \tau_{21} + \eta_0 \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21} \]

Example: Solve the Maxwell Model for an expression explicit in the stress tensor

\[ \tau + \frac{\eta_0}{G} \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} \]
First-order, linear differential equations:

\[ \frac{dy}{dx} + y a(x) + b(x) = 0 \]

Integrating function, \( u(x) \)

\[ u(x) = e^{\int a(x')dx'} \]

**Maxwell Model**

(integral version)

\[
\tau(t) = - \int_{-\infty}^{t} \left( \frac{\eta_0}{\lambda} \right) e^{-(t-t')/\lambda} \dot{\gamma}(t') \, dt'
\]

We arrived at this equation following **two different** paths:

- Add up fading contributions of past deformations
- Add viscous and elastic effects in series
What are the predictions of the Maxwell model?

Need to check the predictions to see if what we have done is worth keeping.

Predictions:

• Steady shear
• Steady elongation
• Start-up of steady shear
• Step shear strain
• Small-amplitude oscillatory shear
**Steady Shear Flow Material Functions**

**Kinematics:**

\[ \mathbf{v} \equiv \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix} \]

\[ \dot{\gamma}(t) = \dot{\gamma}_0 = \text{constant} \]

**Material Functions:**

- First normal-stress coefficient: \( \Psi_1 \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} \)
- Second normal-stress coefficient: \( \Psi_2 \equiv \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} \)

Viscosity: \( \eta \equiv \frac{-\tau_{21}}{\dot{\gamma}_0} \)

**Predictions of the (single-mode) Maxwell Model**

\[ \tau + \frac{\eta_0}{G} \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} \]

\[ \tau(t) = -\int_{-\infty}^{t} \left( \frac{\eta_0}{\lambda} \right) e^{-(t-t')/\lambda} \dot{\gamma}(t') \, dt' \]

- **Steady shear**
  - \( \eta = \eta_0 \)
  - \( \Psi_1 = \Psi_2 = 0 \)
  - Fails to predict shear normal stresses.
  - Fails to predict shear-thinning.

- **Steady elongation**
  - \( \tilde{\eta} = 3\eta_0 \)
  - Trouton’s rule
Steady shear viscosity and first normal stress coefficient

There are some systems with a constant viscosity but still start-up effects.

Figure 6.5, p. 173 Binnington and Boger; PIB soln

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Steady shear viscosity and first and second normal stress coefficient

Figure 6.6, p. 174 Magda et al.; PS solns

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Step Shear Strain Material Functions

**Kinematics:**

\[ \hat{\gamma}(t) = \begin{cases} \hat{\gamma}(t)x_2 & \hat{\gamma}(t) \neq 0 \\ 0 & \hat{\gamma}(t) = 0 \end{cases} \]

\[ \hat{\gamma}(t) \rightarrow \lim_{\varepsilon \rightarrow 0} \hat{\gamma}_0 = 0 \leq t < \varepsilon \]

\[ \hat{\gamma}(t) \rightarrow \lim_{\varepsilon \rightarrow 0} \hat{\gamma}_0 = t \geq \varepsilon \]

\[ \hat{\gamma}_0 \varepsilon = \text{constant} = \gamma_0 \]

**Material Functions:**

- **First normal-stress relaxation modulus**
  \[ G(t, \gamma_0) = \frac{-\tau_{11}(t, \gamma_0)}{\gamma_0} \]

- **Second normal-stress relaxation modulus**
  \[ G_{\Psi_2} = \frac{-\tau_{22}(t, \gamma_0)}{\gamma_0} \]

- **Relaxation modulus**
  \[ G_{\Psi_1} = \frac{-\tau_{33}(t, \gamma_0)}{\gamma_0} \]

**Predictions of the (single-mode) Maxwell Model**

\[ \dot{\varepsilon} + \frac{\eta_0}{G} \frac{\partial \varepsilon}{\partial t} = -\eta_0 \dot{\gamma} \]

\[ \varepsilon(t) = \int_{-\infty}^{t} \left( \eta_0 \lambda \right) e^{-t-t'/\lambda} \dot{\gamma}(t') dt' \]

**Shear start-up**

\[ \eta^+(t) = \eta_0 \left( 1 - e^{-t/\lambda} \right) \]

\[ \Psi_1^+(t) = \Psi_2^+(t) = 0 \]

**Step shear strain**

\[ G(t) = \frac{\eta_0}{\lambda} e^{-t/\lambda} \]

\[ G_{\Psi_1} = G_{\Psi_2} = 0 \]
Step-Shear-Strain Material Function $G(t)$ for Maxwell Model

Comparison to experimental data

Figure 8.4, p. 274 data from Einaga et al., PS 20% soln in chlorinated diphenyl

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We can improve this fit by adjusting the Maxwell model to allow multiple relaxation modes

\[ \tau(k) = -\int_{-\infty}^{t} \frac{\eta_k}{\lambda_k} e^{-\frac{(t-t')}{\lambda_k}} \varphi(t') \, dt' \]

\[ \tau(t) = \sum_{k=1}^{N} \tau(k) \]

**Generalized Maxwell Model**

\[ \tau = -\int_{-\infty}^{t} \sum_{k=1}^{3} \frac{\eta_k}{\lambda_k} e^{-\frac{(t-t')}{\lambda_k}} \varphi(t') \, dt' \]

2N parameters (can fit *anything*)

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**Steady Shear Flow Material Functions**

**Kinematics:**

\[ \dot{\gamma} = \begin{pmatrix} \dot{\gamma}(t) \chi_2 \\ 0 \\ 0 \end{pmatrix} \]

\[ \dot{\gamma}(t) = \dot{\gamma}_0 = \text{constant} \]

**Material Functions:**

\[ \eta = -\frac{\tau_{21}}{\dot{\gamma}_0} \]

First normal-stress coefficient

\[ \psi_1 \equiv -\frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0^2} \]

Second normal-stress coefficient

\[ \psi_2 \equiv -\frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0^2} \]
**Step Shear Strain** Material Functions

**Kinematics:**

\[
\varepsilon(t) = \begin{cases} 
    \dot{\varepsilon}(t_2) & t < 0 \\
    0 & 0 \leq t < \varepsilon \\
    0 & t \geq \varepsilon \\
\end{cases}
\]

\[\dot{\varepsilon} = \text{constant} = \gamma_0\]

**Material Functions:**

First normal-stress relaxation modulus

\[G(t, \gamma_0) = \frac{-\tau_{21}(t, \gamma_0)}{\gamma_0}\]

Relaxation modulus

Second normal-stress relaxation modulus

\[G_{\Psi_1} = -\frac{(\tau_{11} - \tau_{22})}{\gamma_0^2}\]

\[G_{\Psi_2} = -\frac{(\tau_{22} - \tau_{33})}{\gamma_0^2}\]

Predictions of the Generalized Maxwell Model

\[\tau = -\int^t_{\infty} \left[ \sum_{k=1}^N \eta_k e^{-(t-t')/\lambda_k} \right] \dot{\gamma}(t') \, dt'\]

**Steady shear**

\[\eta = \sum_{k=1}^N \eta_k \quad \text{Fails to predict shear normal stresses}\]

\[\Psi_1 = \Psi_2 = 0 \quad \text{Fails to predict shear-thinning}\]

**Step shear strain**

\[G(t) = \sum_{k=1}^N \frac{\eta_k}{\lambda_k} e^{-t/\lambda_k}\]

\[G_{\Psi_1} = G_{\Psi_2} = 0 \quad \text{This function can fit any observed data; note that the GMM does not predict shear normal stresses.}\]
Fitting $G(t)$ to Generalized Maxwell Model

Figure 8.4, p. 274 data from Einaga et al., PS 20% soln in chlorinated diphenyl

The Linear-Viscoelastic Models

Differential Maxwell (one mode):\[ \tau + \eta_0 \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} \]

Integral Maxwell (one mode):\[ \tau = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{-\frac{(t-t')}{\lambda}} \dot{\gamma}(t') dt' \]

Generalized Maxwell model (N modes):\[ \tau = -\int_{-\infty}^{t} \left[ \sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{-\frac{(t-t')}{\lambda_k}} \right] \dot{\gamma}(t') dt' \]
The Linear-Viscoelastic Models

Differential Maxwell (one mode):
\[ \tau + \frac{\eta_0}{G} \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} \]

Integral Maxwell (one mode):
\[ \tau = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{-\frac{(t-t')}{\lambda}} \dot{\gamma}(t') \, dt' \]

Generalized Maxwell model (N modes):
\[ \tau = -\int_{-\infty}^{t} \sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{-\frac{(t-t')}{\lambda_k}} \dot{\gamma}(t') \, dt' \]

Since the term in brackets is just the predicted relaxation modulus \( G(t) \), we can write an even more general linear viscoelastic model by leaving this function unspecified.

Generalized Linear-Viscoelastic Model:
\[ \tau = -\int_{-\infty}^{t} G(t-t') \dot{\gamma}(t') \, dt' \]
**Small-Amplitude Oscillatory Shear** Material Functions

**Kinematics:**
\[
\nu = \begin{pmatrix}
\dot{\xi}(t)x_2 \\
0 \\
0
\end{pmatrix}_{123}
\]
\[
\dot{\xi}(t) = \dot{\gamma}_0 \cos \omega t
\]
\[
\gamma_0 = \frac{\dot{\gamma}_0}{\omega}
\]

**Material Functions:**
\[
\frac{-\tau_{23}(t, \gamma_0)}{\gamma_0} = G' \sin \omega t + G'' \cos \omega t
\]

\[
G'(\omega) = \frac{\tau_0}{\gamma_0} \cos \delta
\]

(\delta is the phase difference between stress and strain)

\[
G''(\omega) = \frac{\tau_0}{\gamma_0} \sin \delta
\]

Storage modulus

Loss modulus

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**Predictions of the Generalized Maxwell Model (GMM) and Generalized Linear-Viscoelastic Model (GLVE)**

Small-amplitude oscillatory shear

GLVE

GMM

\[
G'(\omega) = \frac{\tau_0}{\gamma_0} \cos \delta
\]

\[
G''(\omega) = \frac{\tau_0}{\gamma_0} \sin \delta
\]

\[
G'(\omega) = \omega \int_0^\infty G(s) \cos \omega s \, ds
\]

\[
G''(\omega) = \omega \int_0^\infty G(s) \sin \omega s \, ds
\]

\[
G'(\omega) = \sum_{k=1}^N \frac{\eta_k \lambda_k \omega^2}{1 + \lambda_k \omega^2}
\]

\[
G''(\omega) = \sum_{k=1}^N \frac{\eta_k \omega}{1 + \lambda_k \omega^2}
\]

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Predictions of (single-mode) Maxwell Model in SAOS

\[ G'(\omega) = \frac{g_1 \lambda_1 \omega^2}{1 + (\lambda_1 \omega)^2} = \frac{\eta \omega}{1 + (\lambda_1 \omega)^2} \]

\[ G''(\omega) = \frac{g_1 \lambda_1 \omega}{1 + (\lambda_1 \omega)^2} = \frac{\eta \omega}{1 + (\lambda_1 \omega)^2} \]

Predictions of (multi-mode) Maxwell Model in SAOS

\[ G'(\omega) = \sum_{i=1}^{N} \frac{\eta_i \lambda_i \omega^2}{1 + (\lambda_i \omega)^2} \]

\[ G''(\omega) = \sum_{i=1}^{N} \frac{\eta_i \omega}{1 + (\lambda_i \omega)^2} \]
Limitations of the GLVE Models

- Predicts constant shear viscosity
- Only valid in regime where strain is additive (small-strain, low rates)
- All stresses are proportional to the deformation-rate tensor; thus shear normal stresses cannot be predicted
- Cannot describe flows with a superposed rigid rotation (as we will now discuss; see Morrison p296)
Steady shear viscosity and first and second normal stress coefficient

EXAMPLE: Drag flow of a Generalized Linear-Viscoelastic fluid between infinite parallel plates

*steady state
*incompressible fluid
*infinitely wide, long

Figure 6.6, p. 174 Magda et al.; PS solns

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Shear flow in a rotating frame of reference

\[ \begin{align*}
  a &= (x - x_o) \cos \Omega t \\
  b &= (x - x_o) \sin \Omega t \\
  c &= (y - y_o) \sin \Omega t \\
  d + b &= (y - y_o) \cos \Omega t
\end{align*} \]

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Summary: Generalized Linear-Viscoelastic Constitutive Equations

**PRO:**
- A first constitutive equation with memory
- Can match SAOS, step-strain data very well
- Captures start-up/cessation effects
- Simple to calculate with
- Can be used to calculate the LVE spectrum

**CON:**
- Fails to predict shear normal stresses
- Fails to predict shear-thinning/thickening
- Only valid at small strains, small rates
- Not frame-invariant