

C.2 Differential Operations in Curvilinear Coordinates

TABLE C.3
Differential Operations in the Cylindrical Coordinate System r, θ, z

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_z \end{pmatrix}_{r\theta z}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r} \frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial a}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla \cdot \underline{w} = \frac{1}{r} \frac{\partial}{\partial r} (r w_r) + \frac{1}{r} \frac{\partial w_\theta}{\partial \theta} + \frac{\partial w_z}{\partial z}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r} \frac{\partial w_z}{\partial \theta} - \frac{\partial w_\theta}{\partial z} \\ \frac{\partial w_r}{\partial z} - \frac{\partial w_z}{\partial r} \\ \frac{1}{r} \frac{\partial (r w_\theta)}{\partial r} - \frac{1}{r} \frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta z}$$

$$\underline{A} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{pmatrix}_{r\theta z}$$

$$\nabla \underline{w} = \begin{pmatrix} \frac{\partial w_r}{\partial r} & \frac{\partial w_\theta}{\partial r} & \frac{\partial w_z}{\partial r} \\ \frac{1}{r} \frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} & \frac{1}{r} \frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} & \frac{1}{r} \frac{\partial w_z}{\partial \theta} \\ \frac{\partial w_r}{\partial z} & \frac{\partial w_\theta}{\partial z} & \frac{\partial w_z}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial (r w_r)}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 w_r}{\partial \theta^2} + \frac{\partial^2 w_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial w_\theta}{\partial \theta} \\ \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial (r w_\theta)}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 w_\theta}{\partial \theta^2} + \frac{\partial^2 w_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial w_r}{\partial \theta} \\ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 w_z}{\partial \theta^2} + \frac{\partial^2 w_z}{\partial z^2} \end{pmatrix}_{r\theta z}$$

$$\nabla \cdot \underline{A} = \begin{pmatrix} \frac{1}{r} \frac{\partial}{\partial r} (r A_{rr}) + \frac{1}{r} \frac{\partial A_{\theta r}}{\partial \theta} + \frac{\partial A_{zr}}{\partial z} - \frac{A_{\theta\theta}}{r} \\ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_{r\theta}) + \frac{1}{r} \frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\partial A_{z\theta}}{\partial z} + \frac{A_{\theta r} - A_{r\theta}}{r} \\ \frac{1}{r} \frac{\partial}{\partial r} (r A_{rz}) + \frac{1}{r} \frac{\partial A_{\theta z}}{\partial \theta} + \frac{\partial A_{zz}}{\partial z} \end{pmatrix}_{r\theta z} \quad (\text{C.3-10})$$

$$\underline{u} \cdot \nabla \underline{w} = \begin{pmatrix} u_r \left(\frac{\partial w_r}{\partial r} \right) + u_\theta \left(\frac{1}{r} \frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} \right) + u_z \left(\frac{\partial w_r}{\partial z} \right) \\ u_r \left(\frac{\partial w_\theta}{\partial r} \right) + u_\theta \left(\frac{1}{r} \frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} \right) + u_z \left(\frac{\partial w_\theta}{\partial z} \right) \\ u_r \left(\frac{\partial w_z}{\partial r} \right) + u_\theta \left(\frac{1}{r} \frac{\partial w_z}{\partial \theta} \right) + u_z \left(\frac{\partial w_z}{\partial z} \right) \end{pmatrix}_{r\theta z} \quad (\text{C.3-11})$$

TABLE C.4
Differential Operations in the Spherical Coordinate System r, θ, ϕ

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_\phi \end{pmatrix}_{r\theta\phi} \quad (\text{C.4-1})$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \quad (\text{C.4-2})$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r} \frac{\partial a}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial a}{\partial \phi} \end{pmatrix}_{r\theta\phi} \quad (\text{C.4-3})$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial a}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial a}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 a}{\partial \phi^2} \quad (\text{C.4-4})$$

$$\nabla \cdot \underline{w} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 w_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (w_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial w_\phi}{\partial \phi} \quad (\text{C.4-5})$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (w_\phi \sin \theta) - \frac{1}{r \sin \theta} \frac{\partial w_\theta}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial w_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r w_\phi) \\ \frac{1}{r} \frac{\partial}{\partial r} (r w_\theta) - \frac{1}{r} \frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta\phi} \quad (\text{C.4-6})$$

$$\underline{A} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{r\phi} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta\phi} \\ A_{\phi r} & A_{\phi\theta} & A_{\phi\phi} \end{pmatrix}_{r\theta\phi} \quad (\text{C.4-7})$$

continued

$$\nabla \underline{w} = \begin{pmatrix} \frac{\partial w_r}{\partial r} & \frac{\partial w_\theta}{\partial r} & \frac{\partial w_\phi}{\partial r} \\ \frac{1}{r} \frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} & \frac{1}{r} \frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} & \frac{1}{r} \frac{\partial w_\phi}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial w_r}{\partial \phi} - \frac{w_\phi}{r} & \frac{1}{r \sin \theta} \frac{\partial w_\theta}{\partial \phi} - \frac{w_\phi}{r} \cot \theta & \frac{1}{r \sin \theta} \frac{\partial w_\phi}{\partial \phi} + \frac{w_\theta}{r} + \frac{w_\phi}{r} \cot \theta \end{pmatrix}_{r\theta\phi} \quad (C.4-8)$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \left\{ \frac{\partial}{\partial r} \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 w_r) \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial w_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 w_r}{\partial \phi^2} \right. \\ \left. - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (w_\theta \sin \theta) - \frac{2}{r^2 \sin \theta} \frac{\partial w_\phi}{\partial \phi} \right\} \\ \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial w_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (w_\theta \sin \theta) \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 w_\theta}{\partial \phi^2} \right. \\ \left. + \frac{2}{r^2} \frac{\partial w_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial w_\phi}{\partial \phi} \right\} \\ \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial w_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (w_\phi \sin \theta) \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 w_\phi}{\partial \phi^2} \right. \\ \left. + \frac{2}{r^2 \sin \theta} \frac{\partial w_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial w_\theta}{\partial \phi} \right\} \end{pmatrix}_{r\theta\phi} \quad (C.4-9)$$

$$\nabla \cdot \underline{A} = \begin{pmatrix} \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_{\theta r} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_{\phi r}}{\partial \phi} - \frac{A_{\theta\theta} + A_{\phi\phi}}{r} \\ \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 A_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_{\phi\theta}}{\partial \phi} + \frac{(A_{\theta r} - A_{r\theta}) - A_{\phi\phi} \cot \theta}{r} \\ \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 A_{r\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_{\theta\phi} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_{\phi\phi}}{\partial \phi} + \frac{(A_{\phi r} - A_{r\phi}) + A_{\phi\theta} \cot \theta}{r} \end{pmatrix}_{r\theta\phi} \quad (C.4-10)$$

$$\underline{u} \cdot \nabla \underline{w} = \begin{pmatrix} u_r \left(\frac{\partial w_r}{\partial r} \right) + u_\theta \left(\frac{1}{r} \frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} \right) + u_\phi \left(\frac{1}{r \sin \theta} \frac{\partial w_r}{\partial \phi} - \frac{w_\phi}{r} \right) \\ u_r \left(\frac{\partial w_\theta}{\partial r} \right) + u_\theta \left(\frac{1}{r} \frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} \right) + u_\phi \left(\frac{1}{r \sin \theta} \frac{\partial w_\theta}{\partial \phi} - \frac{w_\phi}{r} \cot \theta \right) \\ u_r \left(\frac{\partial w_\phi}{\partial r} \right) + u_\theta \left(\frac{1}{r} \frac{\partial w_\phi}{\partial \theta} \right) + u_\phi \left(\frac{1}{r \sin \theta} \frac{\partial w_\phi}{\partial \phi} + \frac{w_r}{r} + \frac{w_\theta}{r} \cot \theta \right) \end{pmatrix}_{r\theta\phi} \quad (C.4-11)$$

TABLE C.5
Continuity Equation in Three Coordinate Systems

Continuity equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0 \quad (C.5-1)$$

continued

Continuity equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \quad (\text{C.5-2})$$

Continuity equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\phi)}{\partial \phi} = 0 \quad (\text{C.5-3})$$

TABLE C.6
Equation of Motion for Incompressible Fluids in Three Coordinate Systems

Equation of motion for incompressible fluid, Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \rho g_x \quad (\text{C.6-1})$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \rho g_y \quad (\text{C.6-2})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho g_z \quad (\text{C.6-3})$$

Equation of motion for incompressible fluid, cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r} \frac{\partial(r \tau_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} \right] + \rho g_r \quad (\text{C.6-4})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} - \left[\frac{1}{r^2} \frac{\partial(r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} \right] + \rho g_\theta \quad (\text{C.6-5})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r} \frac{\partial(r \tau_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \right] + \rho g_z \quad (\text{C.6-6})$$

Equation of motion for incompressible fluid, spherical coordinates

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\ &= -\frac{\partial p}{\partial r} - \left[\frac{1}{r^2} \frac{\partial(r^2 \tau_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tau_{r\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] + \rho g_r \end{aligned} \quad (\text{C.6-7})$$

$$\begin{aligned} & \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) \\ &= -\frac{1}{r} \frac{\partial p}{\partial \theta} - \left[\frac{1}{r^2} \frac{\partial(r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tau_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{(\cot \theta) \tau_{\phi\phi}}{r} \right] + \rho g_\theta \end{aligned} \quad (\text{C.6-8})$$

$$\begin{aligned} & \rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} - \frac{v_\theta v_\phi \cot \theta}{r} \right) \\ &= -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} - \left[\frac{1}{r^2} \frac{\partial(r^2 \tau_{r\phi})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} - \frac{(2 \cot \theta) \tau_{\theta\phi}}{r} \right] + \rho g_\phi \end{aligned} \quad (\text{C.6-9})$$