5. continuity:

$$\nabla \cdot \underline{v} = 0 \qquad \underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\frac{\partial v_1}{\partial x_1} + \cancel{\frac{\partial v_2}{\partial x_2}} + \cancel{\frac{\partial v_3}{\partial x_3}} = 0$$

$$\boxed{\frac{\partial v_1}{\partial x_1} = 0}$$

EOM: $$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\underline{v} \cdot \nabla \underline{v}}}_{\text{unidir}}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \underbrace{\cancel{\rho \underline{g}}}_{0}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} -\frac{\partial p}{\partial x_1} \\ -\frac{\partial p}{\partial x_2} \\ -\frac{\partial p}{\partial x_3} \end{pmatrix}_{123} - \begin{pmatrix} (\nabla \cdot \underline{\underline{\tau}})_1 \\ (\nabla \cdot \underline{\underline{\tau}})_2 \\ (\nabla \cdot \underline{\underline{\tau}})_3 \end{pmatrix}_{123}$$

$$\nabla\cdot\underline{\underline{\tau}} = \frac{\partial}{\partial x_i}\overbrace{\hat{e}_i\cdot\tau_{mp}\hat{e}_m}^{\delta_{im}}\hat{e}_p$$

"i becomes m"

$$= \frac{\partial \tau_{mp}}{\partial x_m}\hat{e}_p$$

$$= \left(\frac{\partial \tau_{1p}}{\partial x_1} + \frac{\partial \tau_{2p}}{\partial x_2} + \frac{\partial \tau_{3p}}{\partial x_3}\right)\hat{e}_p$$

$$= \begin{pmatrix} \cancel{\frac{\partial \tau_{11}}{\partial x_1}} + \frac{\partial \tau_{21}}{\partial x_2} + \cancel{\frac{\partial \tau_{31}}{\partial x_3}} \\ \frac{\partial \tau_{12}}{\partial x_1} + \cancel{\frac{\partial \tau_{22}}{\partial x_2}} + \cancel{\frac{\partial \tau_{32}}{\partial x_3}} \\ \cancel{\frac{\partial \tau_{13}}{\partial x_1}} + \cancel{\frac{\partial \tau_{23}}{\partial x_2}} + \cancel{\frac{\partial \tau_{33}}{\partial x_3}} \end{pmatrix}_{123}$$

$\underline{\underline{\tau}} = -\eta\dot{\underline{\underline{\gamma}}}$ $\therefore$ cancel terms where $\dot{\gamma}_{ij} = 0$

(see p8)

PL-GNF $\qquad \underline{\underline{\tau}} = -\eta \dot{\underline{\underline{\gamma}}}$

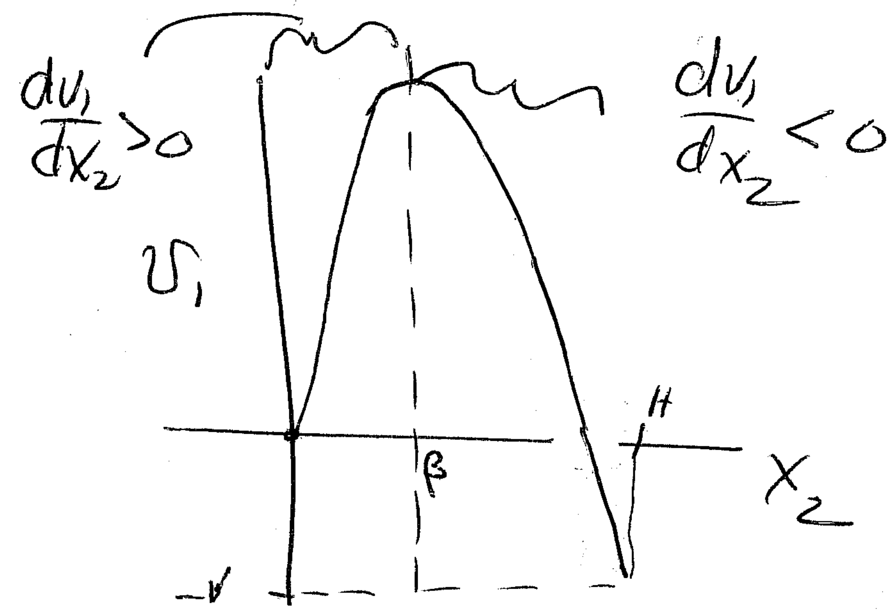
$$\nabla \underline{v} = \begin{pmatrix} \underbrace{\cancel{\frac{\partial v_1}{\partial x_1}}}_{}^{\nabla \cdot \underline{v}} & 0 & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ \cancel{\frac{\partial v_1}{\partial x_3}} & 0 & 0 \end{pmatrix}_{123}$$

wide

$$\dot{\underline{\underline{\gamma}}} = \nabla \underline{v} + (\nabla \underline{v})^T = \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$|\dot{\underline{\underline{\gamma}}}| = \sqrt{\frac{2\left(\frac{\partial v_1}{\partial x_2}\right)^2}{2}} = \pm \frac{\partial v_1}{\partial x_2}$$

Expected sol'n



$0 < x_2 < \beta \qquad \frac{\partial v_1}{\partial x_2} > 0$

$\beta < x_2 < H \qquad \frac{\partial v_1}{\partial x_2} < 0$

split into 2 regimes + solve

EOM

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} -\frac{\partial P}{\partial x_1} \\ -\frac{\partial P}{\partial x_2} \\ -\frac{\partial P}{\partial x_3} \end{pmatrix} - \begin{pmatrix} \frac{\partial \tau_{21}}{\partial x_2} \\ \cancel{\frac{\partial \tau_{12}}{\partial x_1}} \\ 0 \end{pmatrix}_{123}$$

$v_1 \neq$ function of x_1

2-component: $\boxed{\frac{\partial P}{\partial x_2} = 0}$

3-component: $\boxed{\frac{\partial P}{\partial x_3} = 0}$

1-component: $\boxed{\frac{\partial P}{\partial x_1} = -\frac{\partial \tau_{21}}{\partial x_2}}$

$$\underbrace{\frac{\partial P}{\partial x_1}}_{\substack{\text{function} \\ \text{of} \\ x_1 \text{ only}}} = \underbrace{-\frac{\partial \tau_{21}}{\partial x_2}}_{\substack{\text{function of} \\ x_2 \text{ only}}} = \lambda$$

$$\frac{dP}{dx_1} = \lambda$$

$$\boxed{P = \lambda x_1 + C}$$

BC: $x_1 = 0 \quad P = P_0 \Rightarrow \boxed{C = P_0}$

$x_1 = L \quad P = P_L \Rightarrow \boxed{\lambda = \frac{P_L - P_0}{L}}$

$$\boxed{P = \left(\frac{P_L - P_0}{L}\right) x_1 + P_0}$$

$$-\frac{d\tau_{21}}{dx_2} = \frac{P_L - P_0}{L}$$

(c)

$$\boxed{\tau_{21} = -\left(\frac{P_L - P_0}{L}\right)x_2 + C_2}$$

BC on $\underline{v}$: $\boxed{\begin{array}{ll} x_2 = 0 & v_1 = 0 \\ x_2 = H & v_1 = -V \end{array}}$

$$\tau_{21} = -\eta\dot{\gamma}_{21} = -\eta\frac{\partial v_1}{\partial x_2}$$

$$= -m\dot{\gamma}^{n-1}\frac{dv_1}{dx_2}$$

$$\boxed{\tau_{21} = -m\left|\frac{dv_1}{dx_2}\right|^{n-1}\frac{dv_1}{dx_2}}$$

$\leftarrow$ substitute into (c) above, break into two regions + solve!

(B)

NOTE: I DID NOT EXPECT ALL THIS, JUST 2 CASES + A START ON THE SOLN

Bonus

$$\tau_{21} = -\frac{(P_L - P_0)}{L} x_2 + C_2$$

$$\tau_{21} = -m \left| \frac{dv_1}{dx_2} \right|^{n-1} \frac{dv_1}{dx_2}$$

CASE I: $\quad 0 < x_2 < \beta \qquad \frac{dv_1}{dx_2} > 0$

$$\tau_{21} = -m \left(\frac{dv_1}{dx_2} \right)^{n-1} \frac{dv_1}{dx_2}$$

$$\tau_{21} = -m \left(\frac{dv_1}{dx_2} \right)^{n}$$

$$-m \left(\frac{dv_1}{dx_2} \right)^{n} = -\frac{(P_L - P_0)}{L} x_2 + C_2$$

$$\left(\frac{dv_1}{dx_2} \right)^{n} = \frac{P_L - P_0}{mL} x_2 - \frac{C_2}{m}$$

$$\frac{dV_1}{dX_2} = \left(\frac{P_L - P_0}{mL} X_2 - \frac{C_2}{m}\right)^{\frac{1}{n}}$$

integrate:

$$V_1 = \int \left[\left(\frac{P_L - P_0}{mL}\right) X_2 - \frac{C_2}{m}\right]^{\frac{1}{n}} dX_2$$

$$= \int (aX_2 + b)^{\frac{1}{n}} dX_2$$

$$= \frac{1}{a}\int (aX_2 + b)^{\frac{1}{n}} (a\, dX_2)$$

note: $\int u^m du = \frac{u^{m+1}}{m+1}$

$$\boxed{V_1 = \frac{(aX_2 + b)^{\frac{1}{n}+1}}{a\left(\frac{1}{n}+1\right)} + C_3}$$

BC: $X_2 = 0 \quad V_1 = 0$

$$0 = \frac{b^{\frac{1}{n}+1}}{a\left(\frac{1}{n}+1\right)} + C_3$$

$$\boxed{C_3 = -\frac{b^{1/n+1}}{a\left(\frac{1}{n}+1\right)}}$$

$$\boxed{V_1 = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[(aX_2+b)^{\frac{1}{n}+1} - b^{\frac{1}{n}+1}\right]}$$

CASE I

Check: $X_2 = 0$

$V_1 = 0$? yes

$$\boxed{a = \frac{P_L - P_0}{mL} \qquad b = -\frac{C_2}{m}}$$

(negative)

(positive – see p16)

BC: $X_2 = \beta$ $\quad V_1^{\text{case I}} = V_1^{\text{CASE 2}}$

match at max

OR: $X_2 = \beta \quad \frac{dv_1}{dX_2} = 0$

$$\frac{dv_1}{dX_2} = (aX_2 + b)^{\frac{1}{n}}$$

(see p 14)

$$0 = (a\beta + b)^{\frac{1}{n}}$$

$$0 = a\beta + b$$

$$\boxed{\beta = -\frac{b}{a}} = \frac{\frac{C_2}{\cancel{m}}}{\frac{P_L - P_0}{\cancel{m}L}}$$

$$\boxed{\beta = \frac{C_2 L}{P_L - P_0}}$$

still don't know

or

still don't know

$$\boxed{C_2 = \beta\frac{(P_L - P_0)}{L}}$$

note: since $P_L < P_0$ $\boxed{C_2 < 0}$

CASE II

$$\beta < x_2 < H \qquad \frac{dv_1}{dx_2} < 0$$

$$\tau_{21} = -m\left(-\frac{dv_1}{dx_2}\right)^{n-1} \frac{dv_1}{dx_2}$$

$$= m\left(-\frac{dv_1}{dx_2}\right)^{n}$$

← now the neg sign is on the inside

$$m\left(-\frac{dv_1}{dx_2}\right)^{n} = -\frac{(P_L - P_0)}{L} x_2 + C_2$$

Same C_2 both cases (comes from EOM)

$$\left(-\frac{dv_1}{dx_2}\right)^{n} = -\frac{(P_L - P_0)}{mL} x_2 + \frac{C_2}{m}$$

$$-\frac{dv_1}{dx_2} = \left(-\frac{(P_L - P_0)}{mL} x_2 + \frac{C_2}{m}\right)^{\frac{1}{n}}$$

$$-\frac{dV_1}{dx_2} = \left(-ax_2 - b\right)^{\frac{1}{n}}$$

Same definitions of a, b

$$\left.\begin{aligned} u &= -ax_2 - b \\ du &= -a\,dx \end{aligned}\right\} \int u^m du = \frac{u^{m+1}}{m+1}$$

integrate:

$$-V_1 = \left(-\frac{1}{a}\right)\int\left(-ax_2 - b\right)^{\frac{1}{n}}\left(-a\,dx_2\right)$$

$$-V_1 = \frac{\frac{-1}{a}\left(-ax_2 - b\right)^{\frac{1}{n}+1}}{\frac{1}{n}+1} + C_4$$

BC: $x_2 = H \quad V_1 = -V$

$$+V = \frac{-1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH - b\right)^{\frac{1}{n}+1} + C_4$$

$$\boxed{C_4 = V + \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH - b\right)^{\frac{1}{n}+1}}$$

$$V_1 = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)(-ax_2-b)^{\frac{1}{n}+1} - C_4$$ (19)

$$\boxed{V_1 = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[\underset{\text{known}}{(-ax_2-b)}^{\frac{1}{n}+1} - (-aH-b)^{\frac{1}{n}+1}\right] - V}$$

(CASE II)

$b = -\frac{C_2}{n}$ still not known

check: $x_2 = H$

$V_1 = -V$? ✓ yes.

TO calc β or C_2, use BC:

$$x_2 = \beta \qquad \frac{dv_1}{dx_2} = 0$$

From pg 18

$$\frac{dv_1}{dx_2} = -(-ax_2-b)^{\frac{1}{n}}$$

$$0 = -(-a\beta-b)^{\frac{1}{n}}$$

$$0 = (-a\beta-b)^{\frac{1}{n}}$$

$$0 = -a\beta-b$$

$$\beta = -\frac{b}{a}$$ same. (see p12)

Need to use matching condition:

BC: $X_2 = \beta$ $\quad V_1^{\text{CASE I}} = V_1^{\text{CASE II}}$

$$\frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[\cancelto{0}{(a\beta+b)}^{\frac{1}{n}+1} - b^{\frac{1}{n}+1}\right]$$

$$= \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[\cancelto{0}{(-a\beta-b)}^{\frac{1}{n}+1} - (-aH-b)^{\frac{1}{n}+1}\right] - V$$

note: $\beta = -\frac{b}{a}$ (see p16)

$$\cancel{a}\left(-\frac{b}{\cancel{a}}\right) + b = 0$$

$$-a\left(-\frac{b}{a}\right) - b = 0$$

$$\frac{-1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(b^{\frac{1}{n}+1}\right) = \frac{-1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH-b\right)^{\frac{1}{n}+1} - V$$

$$\frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right) b^{\frac{1}{n}+1} = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH-b\right)^{\frac{1}{n}+1} + V$$

$$\frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[b^{\frac{1}{n}+1} - (-aH-b)^{\frac{1}{n}+1}\right] - V = 0$$

$$\underbrace{b = -\frac{C_2}{m} \qquad a = \frac{P_L - P_0}{mL}}$$

this is one eqn for C_2
+ it must be solved
for numerically.

$$\overbrace{\left(\frac{mL}{P_L - P_0}\right)\left(\frac{n}{1+n}\right)\left[\left(-\frac{C_2}{m}\right)^{\frac{1}{n}+1} - \left(\frac{(P_L-P_0)H}{mL} + \frac{C_2}{m}\right)^{\left(\frac{1}{n}\right)+1}\right] - V = 0}$$