CM4650
EXAM 1 12Feb09
Soln

1. $$\underline{\underline{A}} \cdot \underline{g} = A_{ij}\,\hat{e}_i\,\overbrace{\hat{e}_j \cdot g_p\,\hat{e}_p}^{\delta_{jp}\ \text{"j becomes p"}}$$

$$= \boxed{A_{ip}\,g_p\,\hat{e}_i}$$

2. $$\delta_{ni}$$

$$\underline{\underline{M}} : \underline{\underline{M}} = M_{pn}\,\hat{e}_p\,\hat{e}_n : M_{is}\,\hat{e}_i\,\hat{e}_s$$

$$\delta_{ps}$$

$$= M_{si}\,M_{is}$$

$$= (1)(1) + (2)(-1) + \cancel{(0)(2)} + (-1)(2) + (1)(1) + \cancel{(3)(0)} + \cancel{(2)(0)} + \cancel{(0)(3)} + (-2)(-2)$$

$$= 1 - 2 - 2 + 1 + 4 = \boxed{2}$$

3 $\nabla \cdot (\underline{a}\,\underline{d})$

$$= \frac{\partial}{\partial x_i} \underbrace{\hat{e}_i \cdot a_m \hat{e}_m}_{\delta_{im}} d_s \hat{e}_s$$

"i becomes m"

$$= \frac{\partial}{\partial x_m} (a_m d_s) \hat{e}_s$$

$$= \left(a_m \frac{\partial d_s}{\partial x_m} + d_s \frac{\partial a_m}{\partial x_m} \right) \hat{e}_s$$

3-component is when $s = 3$

$$\boxed{\left[\nabla \cdot \underline{a}\,\underline{d}\right]_3 = \sum_{m=1}^{3} \left(a_m \frac{\partial d_3}{\partial x_m} + d_3 \frac{\partial a_m}{\partial x_m} \right)}$$

4.

$$\underline{v} = \begin{pmatrix} -0.5\,\dot{\varepsilon}\,x_1 \\ -0.5\,\dot{\varepsilon}\,x_2 \\ \dot{\varepsilon}\,x_3 \end{pmatrix}_{123}$$

$$\nabla \cdot \underline{v} = \frac{\partial}{\partial x_m} \underbrace{\hat{e}_m \cdot v_p \hat{e}_p}_{\delta_{mp}} = \frac{\partial v_p}{\partial x_p}$$

$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

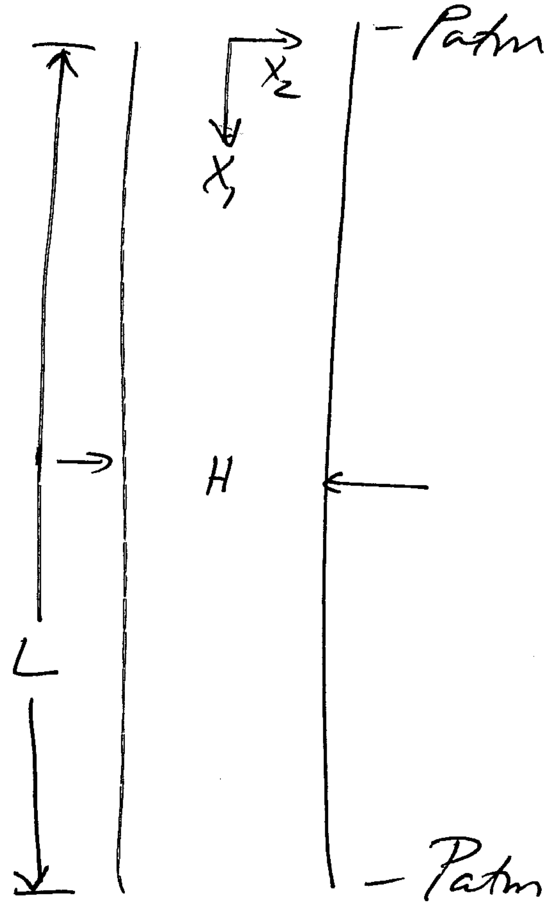
$$= -\frac{\dot{\varepsilon}}{2} + \left(-\frac{\dot{\varepsilon}}{2}\right) + \dot{\varepsilon} = \boxed{0 = \nabla \cdot \underline{v}}$$

$$\nabla \underline{v} = \frac{\partial}{\partial x_s} \hat{e}_s \, v_m \hat{e}_m = \frac{\partial v_m}{\partial x_s} \hat{e}_s \hat{e}_m$$

$$\nabla \underline{v} = \begin{pmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_2}{\partial x_1} & \dfrac{\partial v_3}{\partial x_1} \\[2ex] \dfrac{\partial v_1}{\partial x_2} & \dfrac{\partial v_2}{\partial x_2} & \dfrac{\partial v_3}{\partial x_2} \\[2ex] \dfrac{\partial v_1}{\partial x_3} & \dfrac{\partial v_2}{\partial x_3} & \dfrac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

$$\boxed{\nabla \underline{v} = \begin{pmatrix} -\dfrac{\dot{\varepsilon}}{2} & 0 & 0 \\[2ex] 0 & -\dfrac{\dot{\varepsilon}}{2} & 0 \\[2ex] 0 & 0 & \dot{\varepsilon} \end{pmatrix}_{123}}$$

5.



$$\underline{V} = \begin{pmatrix} V_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\underline{g} = \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}$$

Continuity

$$\nabla \cdot \underline{V} = 0 = \frac{\partial V_1}{\partial x_1} + \cancel{\frac{\partial V_2}{\partial x_2}} + \cancel{\frac{\partial V_3}{\partial x_3}}$$

$$\boxed{\frac{\partial V_1}{\partial x_1} = 0}$$

Navier-Stokes

$$\rho\Big(\underbrace{\cancel{\frac{\partial \underline{V}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{V \cdot \nabla \underline{V}}}_{\text{unidir}}\Big) = -\nabla p + \mu \nabla^2 \underline{V} + \rho \underline{g}$$

$$\nabla^2 \underline{v} = \nabla \cdot \nabla \underline{v}$$

$$= \frac{\partial}{\partial X_s} \underbrace{\hat{e}_s \cdot \frac{\partial}{\partial X_p} \hat{e}_p}_{\delta_{sp}} v_m \hat{e}_m$$

$$= \frac{\partial}{\partial X_p} \frac{\partial}{\partial X_p} v_m \hat{e}_m$$

$\nabla \cdot \underline{v} = 0$

$$\underset{\text{vector}}{\nabla^2 \underline{v}} = \begin{pmatrix} \cancel{\frac{\partial^2 v_1}{\partial X_1^2}} + \frac{\partial^2 v_1}{\partial X_2^2} + \frac{\partial^2 v_1}{\partial X_3^2} \\ \cancel{\frac{\partial^2 v_2}{\partial X_1^2}} + \cancel{\frac{\partial^2 v_2}{\partial X_2^2}} + \cancel{\frac{\partial^2 v_2}{\partial X_3^2}} \\ \cancel{\frac{\partial^2 v_3}{\partial X_1^2}} + \cancel{\frac{\partial^2 v_3}{\partial X_2^2}} + \cancel{\frac{\partial^2 v_3}{\partial X_3^2}} \end{pmatrix}_{123} \qquad \begin{matrix} \\ v_2 = 0 \\ v_3 = 0 \end{matrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} +\frac{\partial P}{\partial x_1} \\ -\frac{\partial P}{\partial x_2} \\ -\frac{\partial P}{\partial x_3} \end{pmatrix}_{123} + \begin{pmatrix} \mu\left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \rho g \\ 0 \\ 0 \end{pmatrix}_{123}$$

2: $\boxed{\frac{\partial P}{\partial x_2} = 0}$

3: $\boxed{\frac{\partial P}{\partial x_3} = 0}$

1: since slit is wide, $\frac{\partial v_1}{\partial x_3} = 0$

since P is same at top + bottom $\frac{\partial P}{\partial x_1} = 0$

$$0 = \mu \frac{d^2 v_1}{dx_2^2} + \rho g$$

$$-\frac{\rho g}{\mu} = \frac{d^2 v_1}{dx_2^2}$$

$$\int d\left(\frac{dv_1}{dx_2}\right) = \int -\frac{\rho g}{\mu} \, dx_2$$

$$\frac{dv_1}{dx_2} = -\frac{\rho g}{\mu} x_2 + C_1$$

$$\int dv_1 = \int \left(-\frac{\rho g}{\mu} x_2 + C_1 \right) dx_2$$

$$\boxed{v_1 = -\frac{\rho g}{\mu} \frac{x_2^2}{2} + C_1 x_2 + C_2}$$

BC:

1. $x_2 = H/2 \quad v_1 = 0$

2. $\begin{cases} x_2 = -H/2 \quad v_1 = 0 \\ \text{or} \\ x_2 = 0 \quad \dfrac{\partial v_1}{\partial x_2} = 0 \end{cases} \Rightarrow \boxed{C_1 = 0}$

BC1: $\quad 0 = -\frac{\rho g}{2\mu}\left(\frac{H}{2}\right)^2 + C_2$

$$C_2 = \frac{\rho g H^2}{8\mu}$$

$$v_1 = -\frac{\rho g}{2\mu} x_2^2 + \frac{\rho g H^2}{8\mu}$$

$$v_1 = \frac{\rho g}{2\mu}\left(\frac{H^2}{4} - x_2^2\right)$$

Flow rate:

$$Q = \int_0^W \int_{-H/2}^{H/2} v_1 \, dx_2 \, dx_3$$

$$= 2W \int_0^{H/2} v_1 \, dx_2$$

$$= \cancel{2}W \frac{\rho g}{\cancel{2}\mu} \int_0^{H/2} \left(\frac{H^2}{4} - x_2^2\right) dx_2$$

$$= \frac{\rho g W}{\mu}\left(\frac{H^2}{4}x_2 - \frac{x_2^3}{3}\right)\Bigg|_0^{H/2}$$

$$\underbrace{\frac{3 \cdot H^3}{3 \cdot 8} - \frac{H^3}{24}}_{\frac{2H^3}{24}}$$

$$\boxed{Q = \frac{\rho g W H^3}{12\mu}}$$