MIDTERM 2
SOLUTION
CM4650
3 MAR 2009

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1. $$\gamma = |\underline{\underline{\dot{\gamma}}}| = \sqrt{\frac{\underline{\underline{\dot{\gamma}}}:\underline{\underline{\dot{\gamma}}}}{2}}$$

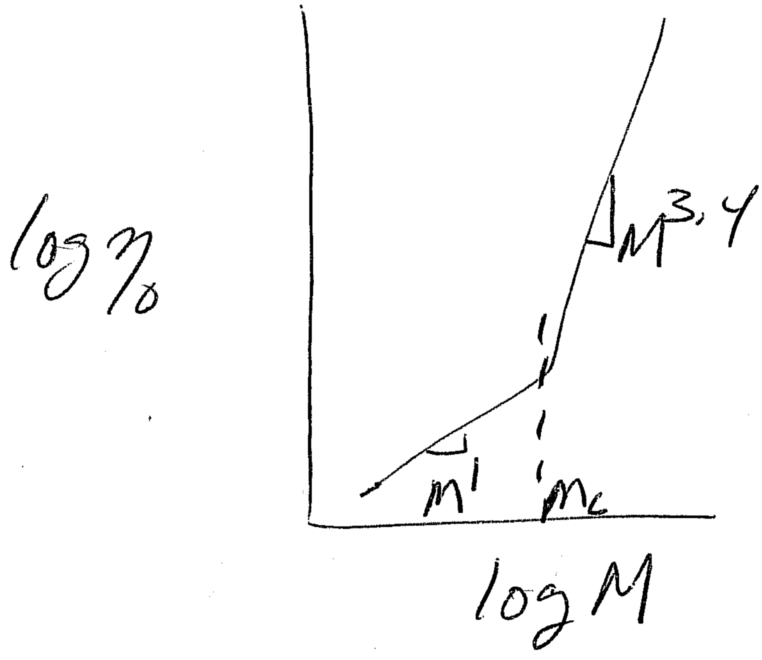
$$= \sqrt{\frac{0 + (6x_2)^2 + 0 + (6x_2)^2 + 0 + 0 + 0 + 0 + 0}{2}}$$

$$= \sqrt{(6x_2)^2} = |6x_2|$$

2. Power-law model - pressure-drop/flow rate, shear thinning

not good: Ψ_1, Ψ_2, time dependence, memory

3.



for $M < M_c$, $\eta_0 \propto M^1$

∴ if we double M, η_0 doubles.

Unless $2M$ is $> M_c$
then η_0 more than doubles. //

4. $\bar{\eta}$? CY-GNF

$$\underline{v} = \begin{pmatrix} -\dot{\varepsilon}_0/2 \; x_1 \\ -\dot{\varepsilon}_0/2 \; x_2 \\ \dot{\varepsilon}_0 \; x_3 \end{pmatrix}_{123}$$

$$\nabla \underline{v} = \begin{pmatrix} -\dot{\varepsilon}_0/2 & 0 & 0 \\ 0 & -\dot{\varepsilon}_0/2 & 0 \\ 0 & 0 & \dot{\varepsilon}_0 \end{pmatrix}_{123}$$

$$\underline{\underline{\dot{\gamma}}} = \nabla \underline{v} + (\nabla \underline{v})^T = \begin{pmatrix} -\dot{\varepsilon}_0 & 0 & 0 \\ 0 & -\dot{\varepsilon}_0 & 0 \\ 0 & 0 & 2\dot{\varepsilon}_0 \end{pmatrix}_{123}$$

GNF: $\underline{\underline{\tau}} = -\eta \underline{\underline{\dot{\gamma}}}$

$$\underline{\underline{\tau}} = \begin{pmatrix} \eta\dot{\varepsilon}_0 & 0 & 0 \\ 0 & \eta\dot{\varepsilon}_0 & 0 \\ 0 & 0 & -2\eta\dot{\varepsilon}_0 \end{pmatrix}_{123}$$

CY GNF:

$$\eta = \eta_\infty + (\eta_0 - \eta_\infty)\left[1+(\dot{\gamma}\lambda)^a\right]^{\frac{n-1}{a}}$$

<u>and $\dot{\gamma}$</u>

$$\dot{\gamma} = |\underline{\dot{\gamma}}| = \sqrt{\frac{\dot{\varepsilon}_0^2 + 0 + 0 + 0 + \dot{\varepsilon}_0^2 + 0 + 0 + 0 + 4\dot{\varepsilon}_0^2}{2}}$$

$$= \sqrt{\frac{6\dot{\varepsilon}_0^2}{2}} = \dot{\varepsilon}_0\sqrt{3}$$

$$\bar{\eta} = \frac{-(\tau_{33}-\tau_{11})}{\dot{\varepsilon}_0}$$

$$= \frac{-(-2\eta\dot{\varepsilon}_0 - \eta\dot{\varepsilon}_0)}{\dot{\varepsilon}_0}$$

$$= \frac{3\eta\cancel{\dot{\varepsilon}_0}}{\cancel{\dot{\varepsilon}_0}} = 3\eta$$

$$\boxed{\bar{\eta} = 3\left(\eta_\infty + (\eta_0-\eta_\infty)\left[1+(\lambda\dot{\varepsilon}_0\sqrt{3})^{a}\right]^{\frac{n-1}{a}}\right)}$$

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5. continuity:

$$\nabla \cdot \underline{v} = 0 \qquad \underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\frac{\partial v_1}{\partial x_1} + \cancelto{}{\frac{\partial v_2}{\partial x_2}} + \cancelto{}{\frac{\partial v_3}{\partial x_3}} = 0$$

$$\boxed{\frac{\partial v_1}{\partial x_1} = 0}$$

EOM:

$$\rho \left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\underline{v} \cdot \nabla \underline{v}}}_{\text{unidir}} \right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \cancelto{0}{\rho \underline{g}}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} -\frac{\partial p}{\partial x_1} \\ -\frac{\partial p}{\partial x_2} \\ -\frac{\partial p}{\partial x_3} \end{pmatrix}_{123} - \begin{pmatrix} (\nabla \cdot \underline{\underline{\tau}})_1 \\ (\nabla \cdot \underline{\underline{\tau}})_2 \\ (\nabla \cdot \underline{\underline{\tau}})_3 \end{pmatrix}_{123}$$

$$\nabla \cdot \underline{\underline{\tau}} = \frac{\partial}{\partial x_i}\overbrace{\hat{e}_i \cdot \tau_{mp}\hat{e}_m}^{\delta_{im}}\hat{e}_p$$

"i becomes m"

$$= \frac{\partial \tau_{mp}}{\partial x_m}\hat{e}_p$$

$$= \left(\frac{\partial \tau_{1p}}{\partial x_1} + \frac{\partial \tau_{2p}}{\partial x_2} + \frac{\partial \tau_{3p}}{\partial x_3}\right)\hat{e}_p$$

$$= \begin{pmatrix} \cancel{\frac{\partial \tau_{11}}{\partial x_1}} + \frac{\partial \tau_{21}}{\partial x_2} + \cancel{\frac{\partial \tau_{31}}{\partial x_3}} \\ \frac{\partial \tau_{12}}{\partial x_1} + \cancel{\frac{\partial \tau_{22}}{\partial x_2}} + \cancel{\frac{\partial \tau_{32}}{\partial x_3}} \\ \cancel{\frac{\partial \tau_{13}}{\partial x_1}} + \cancel{\frac{\partial \tau_{23}}{\partial x_2}} + \cancel{\frac{\partial \tau_{33}}{\partial x_3}} \end{pmatrix}_{123}$$

$\underline{\underline{\tau}} = -\eta \underline{\underline{\dot{\gamma}}}$ $\therefore$ cancel terms where $\dot{\gamma}_{ij} = 0$ (see p8)

PL-GNF

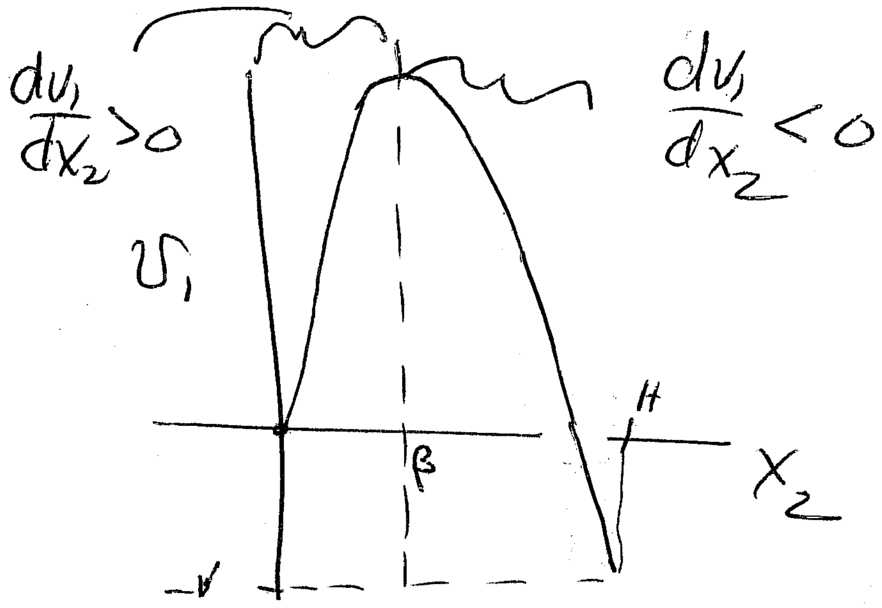
$$\underline{\underline{\tau}} = -\eta \underline{\underline{\dot{\gamma}}}$$

$$\nabla \underline{v} = \begin{pmatrix} \overset{\nabla\cdot\underline{v}}{\cancel{\frac{\partial v_1}{\partial x_1}}} & 0 & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ \underset{\text{wide}}{\cancel{\frac{\partial v_1}{\partial x_3}}} & 0 & 0 \end{pmatrix}_{123}$$

$$\underline{\underline{\dot{\gamma}}} = \nabla \underline{v} + (\nabla \underline{v})^T = \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$|\underline{\underline{\dot{\gamma}}}| = \sqrt{\frac{2\left(\frac{\partial v_1}{\partial x_2}\right)^2}{2}} = \pm \frac{\partial v_1}{\partial x_2}$$

Expected soln



$$\left.\begin{array}{ll} 0 < x_2 < \beta & \frac{\partial v_1}{\partial x_2} > 0 \\ \\ \beta < x_2 < H & \frac{d v_1}{d x_2} < 0 \end{array}\right\} \text{split into 2 regimes + solve}$$

EOM

$$\begin{pmatrix}0\\0\\0\end{pmatrix}_{123} = \begin{pmatrix}-\frac{\partial P}{\partial x_1}\\ -\frac{\partial P}{\partial x_2}\\ -\frac{\partial P}{\partial x_3}\end{pmatrix} - \begin{pmatrix}\frac{\partial \tau_{21}}{\partial x_2}\\ \cancel{\frac{\partial \tau_{12}}{\partial x_1}}\\ 0\end{pmatrix}_{123}$$

$v_1 \neq$ function of x_1

2-component: $\boxed{\frac{\partial P}{\partial x_2} = 0}$

3-component: $\boxed{\frac{\partial P}{\partial x_3} = 0}$

1-component: $\boxed{\frac{\partial P}{\partial x_1} = -\frac{\partial \tau_{21}}{\partial x_2}}$

$$\underbrace{\frac{\partial P}{\partial X_1}}_{\substack{\text{function} \\ \text{of} \\ X_1 \text{ only}}} = \underbrace{-\frac{\partial \tau_{21}}{\partial X_2}}_{\substack{\text{function of} \\ X_2 \text{ only}}} = \lambda$$

$$\frac{dP}{dX_1} = \lambda$$

$$\boxed{P = \lambda X_1 + C}$$

BC: $X_1 = 0 \quad P = P_0 \Rightarrow \boxed{C = P_0}$

$X_1 = L \quad P = P_L \Rightarrow \boxed{\lambda = \frac{P_L - P_0}{L}}$

$$\boxed{P = \left(\frac{P_L - P_0}{L}\right) X_1 + P_0}$$

$$-\frac{d\tau_{21}}{dx_2} = \frac{P_L - P_0}{L}$$

$$\boxed{\tau_{21} = -\left(\frac{P_L - P_0}{L}\right)x_2 + C_2}$$ ⓒ

BC on $\underline{v}$: $\boxed{\begin{array}{ll} x_2 = 0 & v_1 = 0 \\ x_2 = H & v_1 = -V \end{array}}$

$$\tau_{21} = -\eta\dot{\gamma}_{21} = -\eta\frac{\partial v_1}{\partial x_2}$$

$$= -m\dot{\gamma}^{n-1}\frac{dv_1}{dx_2}$$

$$\boxed{\tau_{21} = -m\left|\frac{dv_1}{dx_2}\right|^{n-1}\frac{dv_1}{dx_2}}$$

← substitute into ⓒ above, break into two regions + solve!

☆ NOTE: I DID NOT EXPECT ALL THIS, JUST 2 CASES + A START ON THE SOLN

Ⓑ

BONUS

$$\tau_{21} = -\frac{(P_L - P_0)}{L} x_2 + C_2$$

$$\tau_{21} = -m \left| \frac{dv_1}{dx_2} \right|^{n-1} \frac{dv_1}{dx_2}$$

CASE I: $0 < x_2 < B$ $\quad \frac{dv_1}{dx_2} > 0$

$$\tau_{21} = -m \left(\frac{dv_1}{dx_2} \right)^{n-1} \frac{dv_1}{dx_2}$$

$$\tau_{21} = -m \left(\frac{dv_1}{dx_2} \right)^{n}$$

$$-m \left(\frac{dv_1}{dx_2} \right)^{n} = -\frac{(P_L - P_0)}{L} x_2 + C_2$$

$$\left(\frac{dv_1}{dx_2} \right)^{n} = \frac{P_L - P_0}{mL} x_2 - \frac{C_2}{m}$$

$$\frac{dV_1}{dX_2} = \left(\frac{P_L - P_0}{mL} X_2 - \frac{C_2}{m}\right)^{\frac{1}{n}}$$

integrate:

$$V_1 = \int \left[\left(\frac{P_L - P_0}{mL}\right) X_2 - \frac{C_2}{m}\right]^{\frac{1}{n}} dX_2$$

$$= \int (aX_2 + b)^{\frac{1}{n}} dX_2$$

$$= \frac{1}{a}\int (aX_2 + b)^{\frac{1}{n}} (a\,dX_2)$$

note: $$\int u^m du = \frac{u^{m+1}}{m+1}$$

$$\boxed{V_1 = \frac{(aX_2 + b)^{\frac{1}{n}+1}}{a\left(\frac{1}{n}+1\right)} + C_3}$$

BC: $X_2 = 0$ $\quad V_1 = 0$

$$0 = \frac{b^{\frac{1}{n}+1}}{a\left(\frac{1}{n}+1\right)} + C_3$$

$$\boxed{C_3 = -\frac{b^{\frac{1}{n}+1}}{a\left(\frac{1}{n}+1\right)}}$$

$$\boxed{V_1 = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[(aX_2+b)^{\frac{1}{n}+1} - b^{\frac{1}{n}+1}\right] \quad \text{CASE I}}$$

Check: $X_2 = 0$

$V_1 = 0$? yes

$$\boxed{\begin{aligned} a &= \frac{P_L - P_0}{mL} \\ b &= -\frac{C_2}{m} \end{aligned}}$$

(negative)

(positive – see p16)

BC: $X_2 = \beta \quad V_1^{\text{case I}} = V_1^{\text{CASE 2}}$

↑ match at max

OR: $X_2 = \beta$ $\quad \frac{dv_1}{dX_2} = 0$

$$\frac{dv_1}{dX_2} = (aX_2 + b)^{\frac{1}{n}}$$

(see p14)

$$0 = (a\beta + b)^{\frac{1}{n}}$$

$$0 = a\beta + b$$

$$\boxed{\beta = -\frac{b}{a}} = \frac{\frac{C_2}{\cancel{n}}}{\frac{P_L - P_0}{\cancel{n}L}}$$

$$\boxed{\beta = \frac{C_2 L}{P_L - P_0}}$$

still don't know

or

still don't know

$$\boxed{C_2 = \beta\frac{(P_L - P_0)}{L}}$$

note: since $P_L < P_0$ $\boxed{C_2 < 0}$

CASE II

$$\beta < x_2 < H \qquad \frac{dv_1}{dx_2} < 0$$

$$\tau_{21} = -m\left(-\frac{dv_1}{dx_2}\right)^{n-1}\frac{dv_1}{dx_2}$$

$$= m\left(-\frac{dv_1}{dx_2}\right)^{n}$$

← now the neg sign is on the inside

$$m\left(-\frac{dv_1}{dx_2}\right)^{n} = -\frac{(P_L-P_0)}{L}x_2 + C_2$$

Same C_2 both cases (comes from EOM)

$$\left(-\frac{dv_1}{dx_2}\right)^{n} = -\frac{(P_L-P_0)}{mL}x_2 + \frac{C_2}{m}$$

$$-\frac{dv_1}{dx_2} = \left(-\frac{(P_L-P_0)}{mL}x_2 + \frac{C_2}{m}\right)^{\frac{1}{n}}$$

$$-\frac{dv_1}{dx_2} = \left(-ax_2 - b\right)^{\frac{1}{n}}$$

Same definitions of a, b

$$\left.\begin{array}{l} u = -ax_2 - b \\ du = -a\,dx \end{array}\right\} \int u^m du = \frac{u^{m+1}}{m+1}$$

integrate:

$$-v_1 = \left(-\frac{1}{a}\right)\int\left(-ax_2-b\right)^{\frac{1}{n}}\left(-a\,dx_2\right)$$

$$-v_1 = \frac{\frac{-1}{a}\left(-ax_2-b\right)^{\frac{1}{n}+1}}{\frac{1}{n}+1} + C_4$$

BC: $x_2 = H \quad v_1 = -V$

$$+V = \frac{-1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH-b\right)^{\frac{1}{n}+1} + C_4$$

$$\boxed{C_4 = V + \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH-b\right)^{\frac{1}{n}+1}}$$

$$V_1 = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-ax_2-b\right)^{\frac{1}{n}+1} - C_4$$

$$\boxed{V_1 = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\Big[\big(-\underset{\uparrow\text{known}}{a}x_2-b\big)^{\frac{1}{n}+1} - \big(-aH-\underset{\uparrow}{b}\big)^{\frac{1}{n}+1}\Big] - V}$$

(CASE II)

$b = -C_2/n$ still not known

check: $X_2 = H$

$V_1 = -V$? ✓ yes.

To calc β or C_2, use BC:

$$X_2 = \beta \qquad \frac{dV_1}{dX_2} = 0$$

From pg 18

$$\frac{dV_1}{dX_2} = -\left(-aX_2 - b\right)^{\frac{1}{n}}$$

$$0 = -\left(-a\beta - b\right)^{\frac{1}{n}}$$

$$0 = \left(-a\beta - b\right)^{\frac{1}{n}}$$

$$0 = -a\beta - b$$

$$\beta = -\frac{b}{a}$$ same. (see p12)

Need to use matching condition:

BC: $x_2 = \beta$ $\quad V_1^{CASE\ I} = V_1^{CASE\ II}$

$$\frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[\cancelto{0}{(a\beta+b)}^{\frac{1}{n}+1} - b^{\frac{1}{n}+1}\right]$$

$$= \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[\cancelto{0}{(-a\beta-b)}^{\frac{1}{n}+1} - (-aH-b)^{\frac{1}{n}+1}\right] - V$$

note: $\beta = -\frac{b}{a}$ (see p16)

$$\cancel{a}\left(-\frac{b}{\cancel{a}}\right)+b = 0$$

$$-\cancel{a}\left(-\frac{b}{\cancel{a}}\right)-b = 0$$

$$\frac{-1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(b^{\frac{1}{n}+1}\right) = \frac{-1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH-b\right)^{\frac{1}{n}+1} - V$$

$$\frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right) b^{\frac{1}{n}+1} = \frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left(-aH-b\right)^{\frac{1}{n}+1} + V$$

$$\frac{1}{a}\left(\frac{1}{\frac{1}{n}+1}\right)\left[b^{\frac{1}{n}+1}-(-aH-b)^{\frac{1}{n}+1}\right]-V=0$$

$$\underbrace{b=-\frac{C_2}{m} \qquad a=\frac{P_L-P_0}{mL}}$$

this is one eqn for C_2
+ it must be solved
for numerically.

$$\overbrace{\left(\frac{mL}{P_L-P_0}\right)\left(\frac{n}{1+n}\right)\left[\left(-\frac{C_2}{m}\right)^{\frac{1}{n}+1}-\left(\frac{(P_L-P_0)H}{mL}+\frac{C_2}{m}\right)^{(\frac{1}{n})+1}\right]-V=0}$$