CM4650
FINAL EXAM
SOLUTION
2007

1. a) TRUE
   b) FALSE
   c) TRUE
   d) TRUE

2. $$\nabla(a\underline{v}) = \frac{\partial}{\partial x_p}\hat{e}_p\,(a\,v_j\,\hat{e}_j)$$

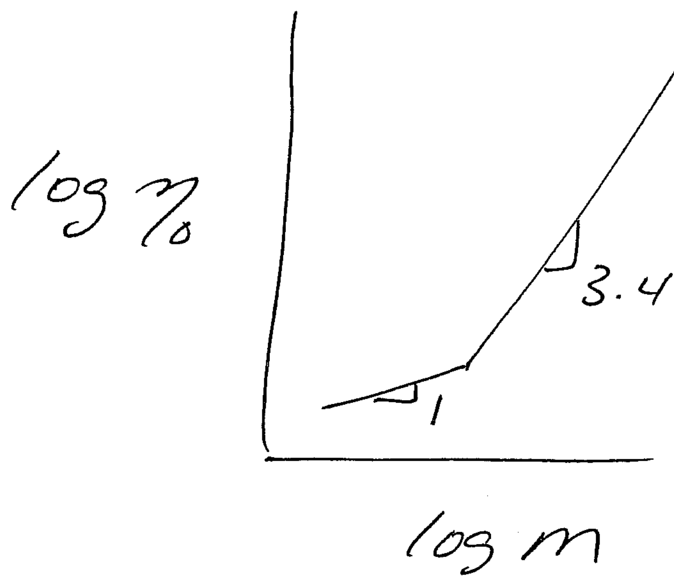
$$= \frac{\partial}{\partial x_p}(a v_j)\,\hat{e}_p\,\hat{e}_j$$

$$= \left(a\,\frac{\partial v_j}{\partial x_p} + v_j\,\frac{\partial a}{\partial x_p}\right)\hat{e}_p\,\hat{e}_j$$

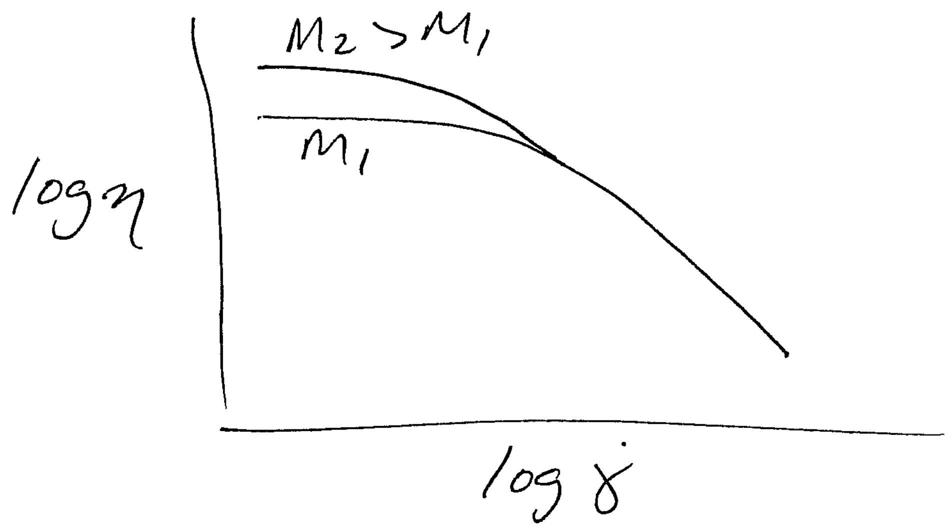
$$= a\,\nabla\underline{v} + (\nabla a)\,\underline{v}$$

3. a) $\eta_0 \propto M \qquad M < M_c$

$\eta \propto M^{3.4} \qquad M > M_c$



b)

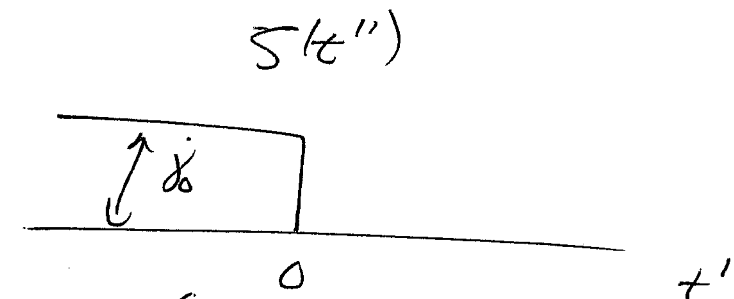


4. $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2}\, e^{-\frac{(t-t')}{\lambda}}\, \underline{\underline{C}}^{-1}(t',t)\, dt'$$

shear flow:

$$\underline{\underline{C}}^{-1} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$$

$$\gamma = \int_{t'}^{t} \dot{\xi}(t'')\, dt''$$



$$t' < 0 \qquad \gamma = \int_{t'}^{0} \dot{\gamma}_0\, dt'' + \underbrace{\cancel{\int_0^t 0\, dt''}}_{0}$$

$$\boxed{t' < 0 \quad \gamma = -t'\dot{\gamma}_0}$$

$$\boxed{t' > 0 \quad \gamma = \int_{t'}^{t} 0\, dt'' = 0}$$

$$\tau_{21} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \gamma \, dt'$$

$$= -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{cases} -t'\dot{\gamma}_0 & t' < 0 \\ 0 & t' \geq 0 \end{cases} dt'$$

$$= -\int_{-\infty}^{0} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} (-t'\dot{\gamma}_0) \, dt'$$

$$\tau_{21} = \int_{-\infty}^{0} \frac{\eta_0 \dot{\gamma}_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} t' \, dt'$$

$$\boxed{\eta^- = \frac{-\tau_{21}}{\dot{\gamma}_0} = \int_{0}^{\infty} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} t' \, dt'}$$

5. $\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$

continuity $\quad \nabla \cdot \underline{v} = \boxed{0 = \frac{\partial v_1}{\partial x_1}}$

EOM: $\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\underline{v} \cdot \nabla \underline{v}}}_{\text{unidirectional}}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \underbrace{\cancel{\rho \underline{g}}}_{\text{neglect gravity}}$

$$\nabla \cdot \underline{\underline{\tau}} = \frac{\partial}{\partial x_i} \underbrace{\hat{e}_i \cdot \tau_{pk} \hat{e}_p}_{\delta_{ip}} \hat{e}_k$$

$$= \frac{\partial \tau_{pk}}{\partial x_p} e_k$$

$$\nabla \cdot \underline{\underline{\tau}} = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \\ \\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} \\ \\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} \end{pmatrix}_{123}$$

$$\underline{\underline{\tau}} = -\eta \underline{\underline{\dot{\gamma}}} = -m \dot{\gamma}^{n-1} \underline{\underline{\dot{\gamma}}}$$

$$\underline{\underline{\dot{\gamma}}} = \nabla \underline{v} + (\nabla \underline{v})^T$$

$\nabla \cdot \underline{v}$

$$\nabla \underline{v} = \begin{pmatrix} \cancel{\frac{\partial v_1}{\partial x_1}} & \cancel{\frac{\partial v_2}{\partial x_1}} & \cancel{\frac{\partial v_3}{\partial x_1}} \\ \\ \frac{\partial v_1}{\partial x_2} & \cancel{\frac{\partial v_2}{\partial x_2}} & \cancel{\frac{\partial v_3}{\partial x_2}} \\ \\ \cancel{\frac{\partial v_1}{\partial x_3}} & \cancel{\frac{\partial v_2}{\partial x_3}} & \cancel{\frac{\partial v_3}{\partial x_3}} \end{pmatrix}_{123} \qquad v_2 = v_3 = 0$$

wide

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$|\underline{\underline{\dot{\gamma}}}| = \sqrt{\left(\frac{\partial v_1}{\partial x_2}\right)^2} = \left|\frac{\partial v_1}{\partial x_2}\right|$$

as x_2 increases, v_1 increases

$$\Rightarrow |\underline{\underline{\dot{\gamma}}}| = \boxed{+\frac{\partial v_1}{\partial x_2} = \dot{\gamma}}$$

$$\underline{\underline{\tau}} = -m\left(\frac{\partial v_1}{\partial x_2}\right)^{n-1} \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$\underline{\underline{\tau}} = \begin{pmatrix} 0 & -m\left(\frac{\partial v_1}{\partial x_2}\right)^n & 0 \\ -m\left(\frac{\partial v_1}{\partial x_2}\right)^n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$\nabla \cdot \underline{\underline{\tau}} = \begin{pmatrix} 0 + \frac{\partial \tau_{21}}{\partial x_2} + 0 \\ \frac{\partial \tau_{12}}{\partial x_1} + 0 + 0 \\ 0 + 0 + 0 \end{pmatrix}_{123}$$

$$\nabla \cdot \underline{\underline{\tau}} = \begin{pmatrix} -\frac{\partial}{\partial x_2}\left(m\left(\frac{\partial v_1}{\partial x_2}\right)^n\right) \\ -\frac{\partial}{\partial x_1}\left(\cancelto{0}{m\left(\frac{\partial v_1}{\partial x_2}\right)^n}\right) \\ 0 \end{pmatrix}_{123} \qquad v_1 \neq f(x_1)$$

EOM: $0 = -\nabla p - \nabla \cdot \underline{\underline{\tau}}$

X_1 - component:

$$\boxed{\frac{\partial P}{\partial X_1} = m\frac{\partial}{\partial X_2}\left(\frac{\partial v_1}{\partial X_2}\right)^n}$$ a

X_2 - component:

$$\frac{\partial P}{\partial X_2} = 0$$

X_3 - component:

$$\frac{\partial P}{\partial X_3} = 0$$

SOLVE:

$$\underbrace{\frac{\partial P}{\partial X_1}}_{f(x_1)} = \underbrace{m\frac{\partial}{\partial X_2}\left(\frac{\partial v_1}{\partial X_2}\right)^n}_{g(x_2)} = \lambda$$

$$\frac{dP}{dX_1} = \lambda$$

$$\boxed{P = \lambda X_1 + C_1}$$

BC: $X_1 = 0 \quad P = P_0$
$X_1 = L \quad P = P_L$

$$\Rightarrow \boxed{P = \frac{P_L - P_0}{L} x_1 + P_0}$$

$$\lambda = \frac{P_L - P_0}{L}$$

$$c_1 = P_0$$

$$m \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}\right)^n = \frac{P_L - P_0}{L}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}\right)^n = \frac{-(P_0 - P_L)}{mL}$$

$$\left(\frac{\partial v_1}{\partial x_2}\right)^n = \frac{-(P_0 - P_L)}{mL} x_2 + C_2$$

$$\frac{\partial v_1}{\partial x_2} = \left(\frac{-(P_0 - P_L)}{mL} x_2 + C_2\right)^{\frac{1}{n}}$$

$$v_1 = \int \left(-\left(\frac{P_0 - P_L}{mL}\right) x_2 + C_2\right)^{\frac{1}{n}} dx_2 + C_3$$

$$\boxed{v_1 = \frac{mL}{P_0 - P_L}\left(\frac{-(P_0 - P_L)}{mL} x_2 + C_2\right)^{\frac{1}{n}+1}\left(\frac{1}{\frac{1}{n}+1}\right) + C_3}$$

BC: $\boxed{\begin{array}{ll} X_2 = 0 & V_1 = 0 \\ X_2 = H & V_1 = V \end{array}}$ ← use to evaluate c_2, c_3 //