CM4650
FINAL
2 MAY 2008
SOLN

1. The relaxation time is the time it takes for a material to relax after a deformation. In the generalized Maxwell model, there are multiple relaxation times that represent different methods by which polymers relax.

2.
$$\nabla \cdot \underline{\underline{\tau}} = \frac{\partial}{\partial x_i} \overbrace{\hat{e}_i \cdot \tau_{pk} \hat{e}_p}^{\delta_{ip}} \hat{e}_k$$

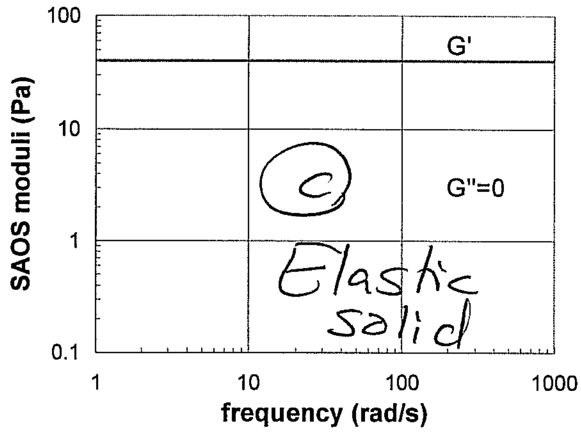
"i becomes p"

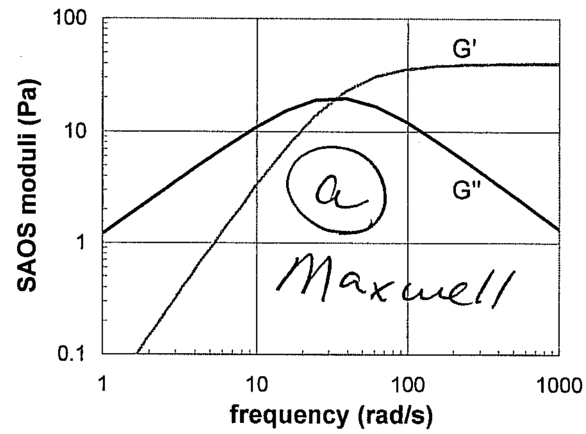
$$\boxed{\nabla \cdot \underline{\underline{\tau}} = \frac{\partial \tau_{pk}}{\partial x_p} e_k}$$

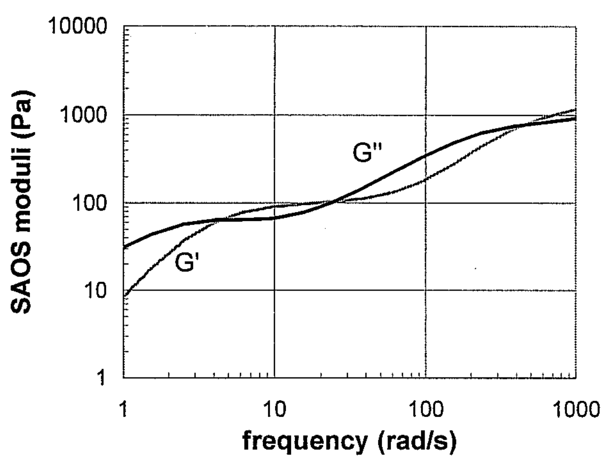
$$\nabla \cdot \underline{\underline{\tau}} = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} \\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} \end{pmatrix}_{123}$$

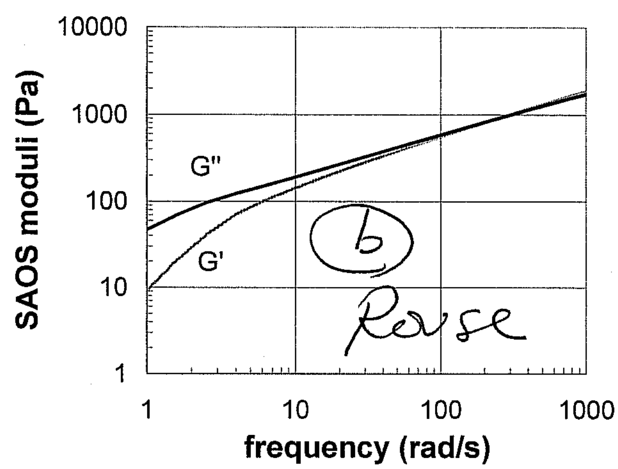
2

3. (20 points) Three sketches of $G'(\omega)$ and $G''(\omega)$ are shown below. Which one is a) a single relaxation-time Maxwell model; b) the Rouse model; and c) an elastic solid. You may write the name on the graph.









4. Poiseuille flow of PL GNF

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{r\theta z} \qquad \nabla \cdot \underline{v} = \frac{\partial v_z}{\partial z} = 0$$

$$\underline{g} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}_{r\theta z}$$

EOM: $$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\underline{v} \cdot \nabla \underline{v}}}_{\text{unidir}}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$$

$$\nabla p = \begin{pmatrix} \frac{\partial P}{\partial r} \\ \frac{1}{r}\frac{\partial P}{\partial \theta} \\ \frac{\partial P}{\partial z} \end{pmatrix}_{r\theta z}$$

PL GNF:

$$\underline{\underline{\tau}} = -\eta \, \underline{\underline{\dot{\gamma}}} \qquad \eta = m \dot{\gamma}^{\,n-1}$$

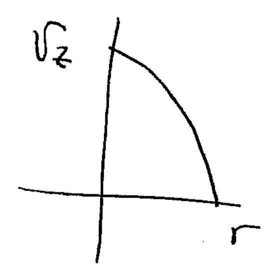
$$\underline{\underline{\dot{\gamma}}} = \nabla \underline{v} + (\nabla \underline{v})^T$$

$$\nabla \underline{v} = \begin{pmatrix} \cancel{\frac{\partial v_r}{\partial r}} & \cancel{\frac{\partial v_\theta}{\partial r}} & \frac{\partial v_z}{\partial r} \\ \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta}{r}} & \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r}{r}} & \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta}} \\ \cancel{\frac{\partial v_r}{\partial z}} & \cancel{\frac{\partial v_\theta}{\partial z}} & \cancel{\frac{\partial v_z}{\partial z}} \end{pmatrix}_{r\theta z}$$

θ symmetry

$\nabla \cdot \underline{v} = 0$

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & 0 & \frac{\partial v_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial v_z}{\partial r} & 0 & 0 \end{pmatrix}_{r\theta z}$$



$$\dot{\gamma} = |\underline{\underline{\dot{\gamma}}}| = \sqrt{\frac{\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}}{2}} = \pm \frac{\partial v_z}{\partial r} = -\frac{\partial v_z}{\partial r} > 0$$

$$\underline{\underline{\tau}} = -m\dot{\gamma}^{n-1}\underline{\underline{\dot{\gamma}}}$$

$$\underline{\underline{\tau}} = \begin{pmatrix} 0 & 0 & m\left(-\frac{\partial v_z}{\partial r}\right)^{n-1}\left(-\frac{\partial v_z}{\partial r}\right) \\ 0 & 0 & 0 \\ m\left(-\frac{\partial v_z}{\partial r}\right)^{n-1}\left(-\frac{\partial v_z}{\partial r}\right) & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$\tau_{zr} = \tau_{rz} \neq 0$$

$$\nabla \cdot \underline{\underline{\tau}} = \text{from Table}$$

no z variation

$$= \begin{pmatrix} \cancel{\frac{\partial \tau_{zr}}{\partial z}} \\ 0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) \end{pmatrix}_{r\theta z}$$

EOM

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{r\theta z} = \begin{pmatrix} -\frac{\partial P}{\partial r} \\ -\frac{1}{r}\frac{\partial P}{\partial \theta} \\ -\frac{\partial P}{\partial z} \end{pmatrix}_{r\theta z} - \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right) \end{pmatrix}_{r\theta z}$$

z-component:

$$\underbrace{\frac{\partial P}{\partial z}}_{\substack{\text{function} \\ \text{of} \\ z}} = \underbrace{-\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\{m\left(-\frac{\partial v_z}{\partial r}\right)^n\right\}\right)}_{\text{function of } r} = \lambda$$

$$\frac{dP}{dz} = \lambda$$

$$P = \lambda z + C_1$$

BC: $z = 0 \quad P = P_0$

$z = L \quad P = P_L$

$$\boxed{P = \frac{P_L - P_0}{L} z + P_0}$$

$$\lambda = \frac{P_L - P_0}{L}$$

$$\frac{d}{dr}\left(mr\left(-\frac{dV_z}{dr}\right)^n\right) = -\lambda r$$

integrating,

$$mr\left(-\frac{dV_z}{dr}\right)^n = \frac{-\lambda r^2}{2} + C_2$$

$$\left(-\frac{dV_z}{dr}\right)^n = \frac{-\lambda}{2m} r + \frac{C_2}{mr}$$

BC: $r=0,\quad \frac{dV_z}{dr} = 0 \Rightarrow \boxed{C_2 = 0}$

$$\left(-\frac{dV_z}{dr}\right)^n = \left(\frac{-\lambda}{2m}\right) r$$

$$-\frac{dV_z}{dr} = \left(\frac{-\lambda}{2m}\right)^{\frac{1}{n}} r^{\frac{1}{n}}$$

integrating,

$$V_z = -\left(\frac{-\lambda}{2m}\right)^{\frac{1}{n}} \frac{r^{\frac{1}{n}+1}}{\frac{1}{n}+1} + C_2$$

BC: $r = R$, $V_z = 0$ $\qquad \lambda = \frac{P_L - P_0}{L}$

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5) What is $\bar{\eta}(\dot{\varepsilon})$ for Lodge model?

Lodge: $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \underline{\underline{C}}^{-1}(t',t)\, dt'$$

elongation: $$\begin{pmatrix} e^{-\epsilon} & 0 & 0 \\ 0 & e^{-\epsilon} & 0 \\ 0 & 0 & e^{2\epsilon} \end{pmatrix}_{123}$$

$$\epsilon = \epsilon(t',t) = \int_{t'}^{t} \dot{\epsilon}(t'')\, dt''$$

$\dot{\epsilon}(t'') = \dot{\epsilon}_0 = \text{constant}$ (steady elongation)

$$\epsilon = \int_{t'}^{t} \dot{\epsilon}_0\, dt'' = \dot{\epsilon}_0 t'' \Big|_{t'}^{t}$$

$$\boxed{\epsilon = (t-t')\dot{\epsilon}_0}$$

⑨

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} e^{-\dot{\epsilon}_0(t-t')} & 0 & 0 \\ 0 & e^{-\dot{\epsilon}_0(t-t')} & 0 \\ 0 & 0 & e^{2\dot{\epsilon}_0(t-t')} \end{pmatrix}_{123} dt'$$

$$\bar{\eta} = \frac{-(\tau_{33} - \tau_{11})}{\dot{\epsilon}_0}$$

$$\tau_{11} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} e^{-\dot{\epsilon}_0(t-t')} dt'$$

let $s = t - t'$

$ds = -dt'$

$t' = -\infty \quad s = \infty$

$t' = t \quad s = 0$

$$\frac{-\lambda^2 \tau_{11}}{\eta_0} = -\int_{\infty}^{0} e^{-\frac{s}{\lambda}} e^{-\dot{\epsilon}_0 s} ds$$

$$= \int_{0}^{\infty} e^{-(\frac{1}{\lambda}+\dot{\epsilon}_0)s} ds$$

$$\int e^u du = e^u$$

$$u = -\left(\frac{1}{\lambda} + \dot{\epsilon}_0\right) s$$

$$du = -\left(\frac{1}{\lambda} + \dot{\epsilon}_0\right) ds$$

$$-\frac{\lambda^2 \tau_{11}}{\eta_0} = \frac{-1}{\frac{1}{\lambda}+\dot{\epsilon}_0} \int_0^\infty e^{-\left(\frac{1}{\lambda} + \dot{\epsilon}_0\right) s} (-)\left(\frac{1}{\lambda} + \dot{\epsilon}_0\right) ds$$

$$= -\frac{1}{\frac{1}{\lambda}+\dot{\epsilon}_0} e^{-\left(\frac{1}{\lambda} + \dot{\epsilon}_0\right) s} \Big|_0^\infty$$

$$= \frac{-1}{\frac{1}{\lambda}+\dot{\epsilon}_0} \left(\cancel{e^{-\infty}} - \underbrace{e^0}_{1}\right)$$

$$-\frac{\lambda^2 \tau_{11}}{\eta_0} = \frac{1}{\frac{1}{\lambda}+\dot{\epsilon}_0}$$

$$\boxed{\tau_{11} = \frac{-\eta_0/\lambda}{1+\dot{\epsilon}_0 \lambda}}$$

$$\tau_{33} = \underbrace{-\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{-\frac{(t-t')}{\lambda}} e^{+2\dot{e}_0 (t-t')} \, dt'}$$

this is the same as the previous integral if $2\dot{e}_0$ is substituted for $-\dot{e}_0$

$$\boxed{\tau_{33} = \frac{-\eta_0/\lambda}{1-2\dot{e}_0\lambda}}$$

$$\frac{-(\tau_{33}-\tau_{11})}{\dot{e}_0} = \bar{\eta} = \left[\left(\frac{\eta_0/\lambda}{1-2\dot{e}_0\lambda}\right) + \left(\frac{-\eta_0/\lambda}{1+\dot{e}_0\lambda}\right)\right]\frac{1}{\dot{e}_0}$$

$$\frac{\bar{\eta}}{\eta_0} = \frac{\cancel{1}+\dot{e}_0\lambda \cancel{-1}+2\dot{e}_0\lambda}{(1+\dot{e}_0\lambda)(1-2\dot{e}_0\lambda)}\frac{1}{\lambda\dot{e}_0}$$

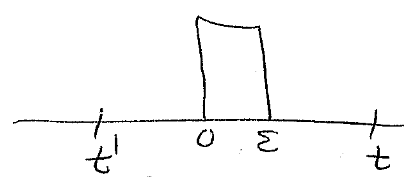
$$\boxed{\frac{\bar{\eta}}{\eta_0} = \frac{3}{(1+\dot{e}_0\lambda)(1-2\dot{e}_0\lambda)}}$$

6. Bonus

$$\dot{\zeta}(t'') = \begin{cases} 0 & t''<0 \\ \frac{\gamma_0}{\varepsilon} & 0<t''<\varepsilon \\ 0 & t''>\varepsilon \end{cases}$$

$$\gamma = \int_{t'}^{t} \dot{\zeta}(t'')\, dt''$$

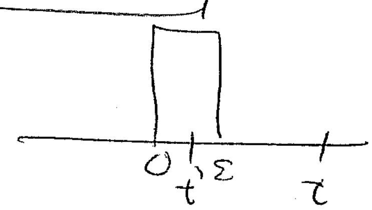
$t'<0$



$$\gamma = \int_{t'}^{0} 0\, dt'' + \int_{0}^{\varepsilon} \frac{\gamma_0}{\varepsilon}\, dt'' + \int_{\varepsilon}^{t} 0\, dt''$$

$$= \frac{\gamma_0}{\varepsilon}(\varepsilon - 0) = \boxed{\gamma_0 \quad t'<0}$$

$0<t'<\varepsilon$



$$\gamma = \int_{t'}^{\varepsilon} \frac{\gamma_0}{\varepsilon}\, dt'' + \int_{\varepsilon}^{t} 0\, dt''$$

$$\gamma = \boxed{\frac{\gamma_0}{\varepsilon}(\varepsilon - t') \qquad 0<t'<\varepsilon}$$

$$\boxed{t''>\varepsilon \qquad \gamma = 0}$$