

## Mathematics Review

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1. Scalar – *a mathematical entity that has magnitude only*

**e.g.:** temperature  $T$   
speed  $v$   
time  $t$   
density  $\rho$

– scalars may be constant or may be variable

### Laws of Algebra for Scalars:

yes commutative	$ab = ba$
yes associative	$a(bc) = (ab)c$
yes distributive	$a(b+c) = ab+ac$

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## Mathematics Review

Polymer Rheology

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2. Vector – *a mathematical entity that has magnitude and direction*

**e.g.:** force on a surface  $f$   
velocity  $v$

– vectors may be constant or may be variable

### Definitions

magnitude of a vector – a scalar associated with a vector

$$|\underline{v}| = v \quad |\underline{f}| = f$$

unit vector – a vector of unit length

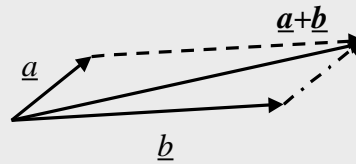
$$\frac{\underline{v}}{|\underline{v}|} = \hat{v}$$

a unit vector in the direction of  $\underline{v}$

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Laws of Algebra for Vectors:

1. Addition



2. Subtraction



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Laws of Algebra for Vectors (continued):

3. Multiplication by scalar  $\alpha \underline{v}$

yes commutative  $\alpha \underline{v} = \underline{v} \alpha$

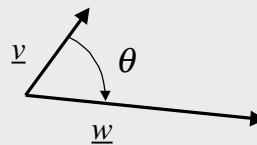
yes associative  $\alpha(\beta \underline{v}) = (\alpha\beta) \underline{v} = \alpha\beta \underline{v}$

yes distributive  $\alpha(\underline{v} + \underline{w}) = \alpha \underline{v} + \alpha \underline{w}$

4. Multiplication of vector by vector

4a. scalar (dot) (inner) product

$$\underline{v} \cdot \underline{w} = vw \cos \theta$$



Note: we can find magnitude with dot product

$$\underline{v} \cdot \underline{v} = vv \cos 0 = v^2$$

$$v = |\underline{v}| = \sqrt{\underline{v} \cdot \underline{v}}$$

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Laws of Algebra for Vectors (continued):

4a. scalar (dot) (inner) product (con't)

yes commutative  $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$

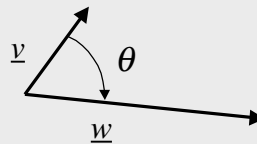
NO associative  ~~$\underline{v} \cdot \underline{w} \cdot \underline{z}$~~  no such operation

yes distributive  $\underline{z} \cdot (\underline{v} + \underline{w}) = \underline{z} \cdot \underline{v} + \underline{z} \cdot \underline{w}$

4b. vector (cross) (outer) product

$$\underline{v} \times \underline{w} = vw \sin \theta \hat{e}$$

$\hat{e}$  is a unit vector perpendicular to both  $\underline{v}$  and  $\underline{w}$  following the right-hand rule



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Laws of Algebra for Vectors (continued):

4b. vector (cross) (outer) product (con't)

NO commutative  $\underline{v} \times \underline{w} \neq \underline{w} \times \underline{v}$

yes associative  $\underline{v} \times \underline{w} \times \underline{z} = (\underline{v} \times \underline{w}) \times \underline{z} = \underline{v} \times (\underline{w} \times \underline{z})$

yes distributive  $\underline{z} \times (\underline{v} + \underline{w}) = (\underline{z} \times \underline{v}) + (\underline{z} \times \underline{w})$

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## Coordinate Systems

- Allow us to make actual calculations with vectors

Rule: any three vectors that are *non-zero* and *linearly independent* (non-coplanar) may form a coordinate basis

Three vectors are linearly dependent if  $\alpha$ ,  $\beta$ , and  $\gamma$  can be found such that:

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} = \underline{0}$$

for  $\alpha, \beta, \gamma \neq 0$

If  $\alpha$ ,  $\beta$ , and  $\gamma$  are found to be zero, the vectors are linearly independent.

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## How can we do actual calculations with vectors?

Rule: *any* vector may be expressed as the linear combination of three, non-zero, non-coplanar basis vectors

any vector  $\underline{a}$  =  $a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}_{xyz}$

coefficient of  $\underline{a}$  in the  $\hat{e}_y$  direction

$$= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

$$= \sum_{j=1}^3 a_j \hat{e}_j$$

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Trial calculation: dot product of two vectors

$$\begin{aligned}
 \underline{a} \cdot \underline{b} &= (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) \\
 &= a_1 \hat{e}_1 \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) + \\
 &\quad a_2 \hat{e}_2 \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) + \\
 &\quad a_3 \hat{e}_3 \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) \\
 &= a_1 \hat{e}_1 \cdot b_1 \hat{e}_1 + a_1 \hat{e}_1 \cdot b_2 \hat{e}_2 + a_1 \hat{e}_1 \cdot b_3 \hat{e}_3 + \\
 &\quad a_2 \hat{e}_2 \cdot b_1 \hat{e}_1 + a_2 \hat{e}_2 \cdot b_2 \hat{e}_2 + a_2 \hat{e}_2 \cdot b_3 \hat{e}_3 + \\
 &\quad a_3 \hat{e}_3 \cdot b_1 \hat{e}_1 + a_3 \hat{e}_3 \cdot b_2 \hat{e}_2 + a_3 \hat{e}_3 \cdot b_3 \hat{e}_3
 \end{aligned}$$

If we choose the basis to be orthonormal - mutually perpendicular and of unit length - then we can simplify.

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If we choose the basis to be orthonormal - mutually perpendicular and of unit length, then we can simplify.

$$\begin{aligned}
 \hat{e}_1 \cdot \hat{e}_1 &= 1 \\
 \hat{e}_1 \cdot \hat{e}_2 &= 0 \\
 \hat{e}_1 \cdot \hat{e}_3 &= 0 \\
 &\dots
 \end{aligned}$$

$$\begin{aligned}
 \underline{a} \cdot \underline{b} &= a_1 \hat{e}_1 \cdot b_1 \hat{e}_1 + a_1 \hat{e}_1 \cdot b_2 \hat{e}_2 + a_1 \hat{e}_1 \cdot b_3 \hat{e}_3 + \\
 &\quad a_2 \hat{e}_2 \cdot b_1 \hat{e}_1 + a_2 \hat{e}_2 \cdot b_2 \hat{e}_2 + a_2 \hat{e}_2 \cdot b_3 \hat{e}_3 + \\
 &\quad a_3 \hat{e}_3 \cdot b_1 \hat{e}_1 + a_3 \hat{e}_3 \cdot b_2 \hat{e}_2 + a_3 \hat{e}_3 \cdot b_3 \hat{e}_3 \\
 &= a_1 b_1 + a_2 b_2 + a_3 b_3
 \end{aligned}$$

We can generalize this operation with a technique called Einstein notation.

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## Einstein Notation

a system of notation for vectors and tensors that allows for the calculation of results in Cartesian coordinate systems.

$$\begin{aligned} \mathbf{a} &= a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 \\ &= \sum_{j=1}^3 a_j\hat{e}_j \\ &= a_j\hat{e}_j = a_m\hat{e}_m \end{aligned}$$

- the initial choice of subscript letter is *arbitrary*
- the presence of a pair of like subscripts implies a missing summation sign

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## Einstein Notation (con't)

The result of the dot products of basis vectors can be summarized by the Kronecker delta function

$$\begin{aligned} \hat{e}_1 \cdot \hat{e}_1 &= 1 \\ \hat{e}_1 \cdot \hat{e}_2 &= 0 \\ \hat{e}_1 \cdot \hat{e}_3 &= 0 \\ \dots & \end{aligned} \quad \hat{e}_i \cdot \hat{e}_p = \delta_{ip} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases}$$

Kronecker delta

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Einstein Notation (con't)

To carry out a dot product of two arbitrary vectors . . .

Detailed Notation	Einstein Notation
$\begin{aligned} \underline{a} \cdot \underline{b} &= (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) \\ &= a_1 \hat{e}_1 \cdot b_1 \hat{e}_1 + a_1 \hat{e}_1 \cdot b_2 \hat{e}_2 + a_1 \hat{e}_1 \cdot b_3 \hat{e}_3 + \\ &\quad a_2 \hat{e}_2 \cdot b_1 \hat{e}_1 + a_2 \hat{e}_2 \cdot b_2 \hat{e}_2 + a_2 \hat{e}_2 \cdot b_3 \hat{e}_3 + \\ &\quad a_3 \hat{e}_3 \cdot b_1 \hat{e}_1 + a_3 \hat{e}_3 \cdot b_2 \hat{e}_2 + a_3 \hat{e}_3 \cdot b_3 \hat{e}_3 \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$	$\begin{aligned} \underline{a} \cdot \underline{b} &= a_j \hat{e}_j \cdot b_m \hat{e}_m \\ &= a_j \delta_{jm} b_m \\ &= a_j b_j \end{aligned}$