

Summary: Generalized Newtonian Fluid Constitutive Equations

- PRO:**
- A first constitutive equation
 - Can match steady shearing data very well
 - Simple to calculate with
 - Found to predict pressure-drop/flow rate relationships well

- CON:**
- Fails to predict shear normal stresses
 - Fails to predict start-up or cessation effects (time-dependence, memory) – only a function of instantaneous velocity gradient
 - Derived ad hoc from shear observations; unclear of validity in non-shear flows

We now look to address this failing of GNF models by seeking to incorporate memory.

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Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material info})$$

The stress expression:

- *Must be of tensor order*
- *Must be a tensor (independent of coordinate system)*
- *Must be a symmetric tensor*
- *Must make predictions that are independent of the observer*
- *Should correctly predict observed flow/deformation behavior*

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Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f(\underline{\underline{\gamma}}, I_{\underline{\underline{\gamma}}}, II_{\underline{\underline{\gamma}}}, III_{\underline{\underline{\gamma}}}, \dots)$$

Tensor invariants – scalars associated with a tensor that do not depend on coordinate system

The stress expression:

- *Must be of tensor order*
- *Must be a tensor (independent of coordinate system)*
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- *Must make predictions that are independent of the observer*
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Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv \text{trace} \underline{\underline{A}} = \text{tr} \underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

$$\text{trace} \underline{\underline{A}} = \sum_{p=1}^3 A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv \text{trace}(\underline{\underline{A}} \cdot \underline{\underline{A}}) = \underline{\underline{A}} : \underline{\underline{A}} = \sum_{p=1}^3 \sum_{k=1}^3 A_{pk} A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv \text{trace}(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}) = \sum_{p=1}^3 \sum_{j=1}^3 \sum_{h=1}^3 A_{pj} A_{jh} A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.

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Fluids with Memory - Chapter 8

We seek a constitutive equation that includes memory effects.

$$\underline{\underline{\tau}}(t) = f(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material information})$$

calculates the stress at a particular time, t

2 equations so far:

$$\underline{\underline{\tau}}(t) = -\mu \underline{\underline{\dot{\gamma}}}(t)$$

$$\underline{\underline{\tau}}(t) = -\eta(\underline{\underline{\dot{\gamma}}}) \underline{\underline{\dot{\gamma}}}(t) \quad \underline{\underline{\dot{\gamma}}} = \left| \underline{\underline{\dot{\gamma}}} \right|$$

So far, stress at t depends on rate-of-deformation **at t only**

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Current Constitutive Equations

Newtonian $\underline{\underline{\tau}}(t) = -\mu \underline{\underline{\dot{\gamma}}}(t)$

Generalized Newtonian $\underline{\underline{\tau}}(t) = -\eta(\underline{\underline{\dot{\gamma}}}) \underline{\underline{\dot{\gamma}}}(t) \quad \underline{\underline{\dot{\gamma}}} = \left| \underline{\underline{\dot{\gamma}}} \right|$

Neither can predict:

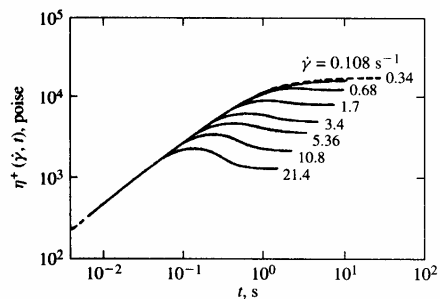
- Shear normal stresses - *this will be wrong so long as we use constitutive equations proportional to $\underline{\underline{\dot{\gamma}}}$*
- stress transients in shear (startup, cessation) - *this flaw seems to be related to omitting fluid memory*

We will try to fix this now; we will address the first point when we discuss advanced constitutive equations

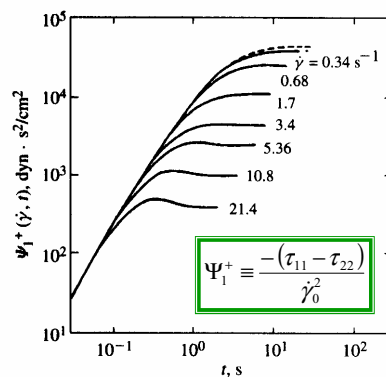
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Startup of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot{\zeta}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \quad \dot{\zeta}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$



$$\eta^+ \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$



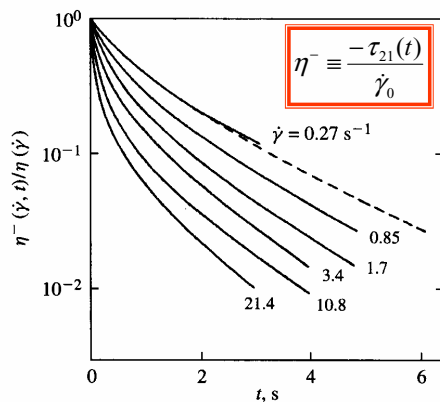
$$\Psi_1^+ \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Figures 6.49, 6.50, p. 208
Menezes and Graessley, PB soln

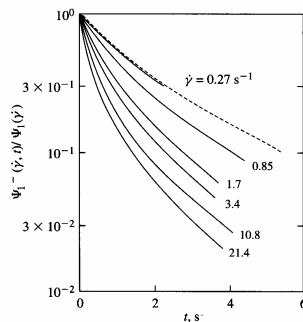
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Cessation of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot{\zeta}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \quad \dot{\zeta}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$$



$$\eta^- \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$



$$\Psi_1^- \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Figures 6.51, 6.52, p. 209 Menezes and
Graessley, PB soln

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How can we incorporate time-dependent effects?

First we explore a simple memory fluid.

Let's construct a new constitutive equation that remembers the stress at a time t_0 seconds ago

$$\underline{\underline{\tau}}(t) = \underbrace{-\tilde{\eta} \underline{\underline{\dot{\gamma}}}(t)}_{\text{Newtonian contribution}} - \underbrace{(0.8\tilde{\eta}) \underline{\underline{\dot{\gamma}}}(t-t_0)}_{\text{contribution from fluid memory}}$$

This is the rate-of-deformation tensor t_0 seconds before time t

$\tilde{\eta}$ is a constant parameter of the model

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What does this model predict?

Steady shear

$$\eta = ?$$

$$\Psi_1 = ?$$

$$\Psi_2 = ?$$

Shear start-up

$$\eta^+(t) = ?$$

$$\Psi_1^+(t) = ?$$

$$\Psi_2^+(t) = ?$$

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Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\zeta}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \quad \dot{\zeta}(t) = \dot{\gamma}_0 = \text{constant}$$

Material Functions:

$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}_0}$$

Viscosity

First normal-stress
coefficient

$$\Psi_1 \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Second normal-
stress coefficient

$$\Psi_2 \equiv \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$$

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Start-up of Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\zeta}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \quad \dot{\zeta}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$

Material Functions:

$$\eta^+ \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$

Shear stress
growth
function

First normal-stress
growth function

$$\Psi_1^+ \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Second normal-
stress growth
function

$$\Psi_2^+ \equiv \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$$

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Cessation of Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\zeta}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \quad \dot{\zeta}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$$

Material Functions:

$$\eta^- \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0} \quad \text{First normal-stress decay function} \quad \Psi_1^- \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

$$\text{Shear stress decay function} \quad \text{Second normal-stress decay function} \quad \Psi_2^- \equiv \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$$

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Predictions of the simple memory fluid

$$\underline{\tau}(t) = -\tilde{\eta} \dot{\gamma}(t) - (0.8\tilde{\eta}) \dot{\gamma}(t - t_0)$$

Steady shear

$$\eta = 1.8\tilde{\eta}$$

$$\Psi_1 = \Psi_2 = 0$$

The steady viscosity reflects contributions from what is currently happening and contributions from what happened t_0 seconds ago.

Shear start-up

$$\eta^+(t) = \begin{cases} 0 & t < 0 \\ \tilde{\eta} & 0 \leq t \leq t_0 \\ 1.8\tilde{\eta} & t \geq t_0 \end{cases}$$

$$\Psi_1^+(t) = \Psi_2^+(t) = 0$$

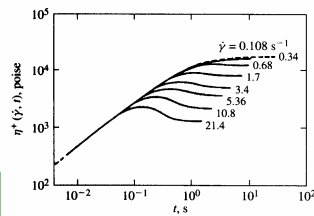
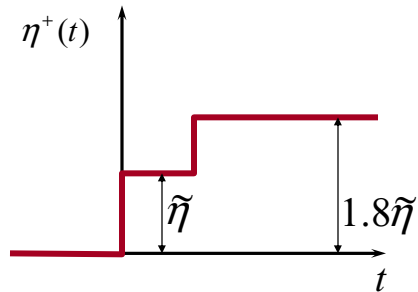
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Predictions of the simple memory fluid

Shear start-up

$$\eta^+(t) = \begin{cases} 0 & t < 0 \\ \tilde{\eta} & 0 \leq t \leq t_0 \\ 1.8\tilde{\eta} & t \geq t_0 \end{cases}$$

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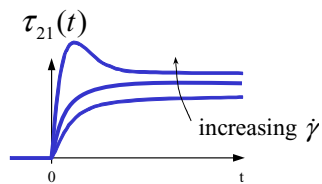


Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB soln

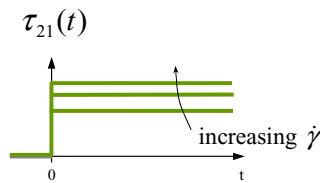
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Predictions of the simple memory fluid Shear start-up

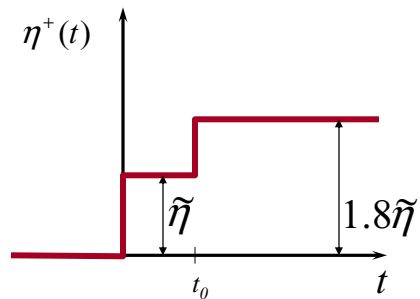
What the data show:



What the GNF models predict:



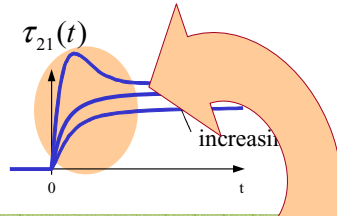
What the simple memory fluid model predict:



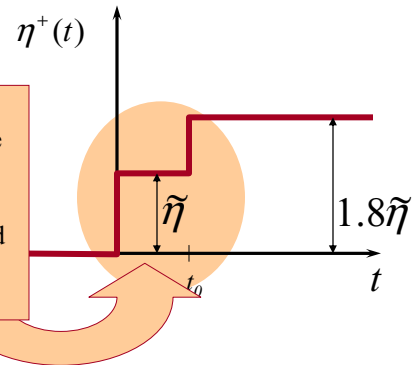
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Predictions of the simple memory fluid Shear start-up

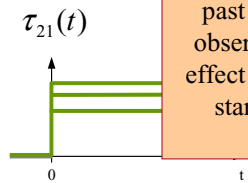
What the data show:



What the simple memory fluid model predict:



What the GNF model predict:



Adding that contribution from the past introduces the observed “build-up” effect to the predicted start-up material functions.

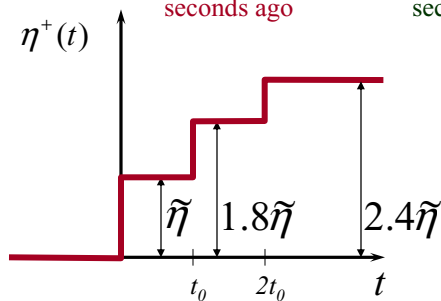
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We can make the stress rise smoother by adding more fading memory terms.

The memory is fading

$$\underline{\underline{\tau}}(t) = \underbrace{-\tilde{\eta} \underline{\underline{\gamma}}(t)}_{\text{Newtonian contribution}} - \underbrace{(0.8\tilde{\eta}) \underline{\underline{\gamma}}(t - t_0)}_{\text{contribution from } t_0 \text{ seconds ago}} - \underbrace{(0.6\tilde{\eta}) \underline{\underline{\gamma}}(t - 2t_0)}_{\text{contribution from } 2t_0 \text{ seconds ago}}$$

Newtonian contribution contribution from t_0 seconds ago contribution from $2t_0$ seconds ago

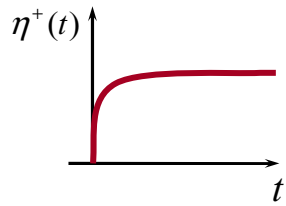


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The fit can be made to be perfectly smooth by using a sum of exponentially decaying terms as the weighting functions.

$$\begin{aligned} \underline{\tau}(t) &= -\tilde{\eta} \left[\underline{\dot{\gamma}}(t) + (0.37) \underline{\dot{\gamma}}(t-t_0) \right. \\ &\quad \left. + (0.14) \underline{\dot{\gamma}}(t-2t_0) + (0.05) \underline{\dot{\gamma}}(t-3t_0) + \dots \right] \\ &= -\tilde{\eta} \sum_{p=0}^{\infty} e^{-p} \underline{\dot{\gamma}}(t-pt_0) \end{aligned}$$

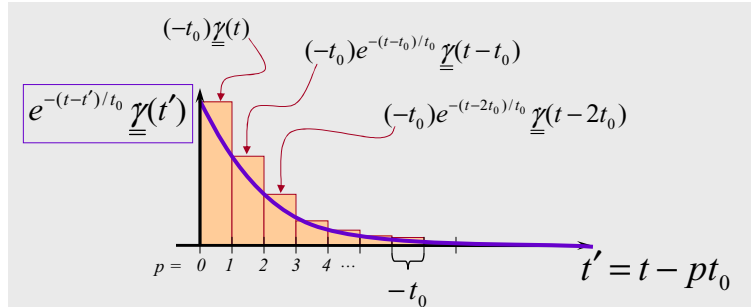
p	e^{-p}
0	1.00
1	0.37
2	0.14
3	0.05
4	0.02



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This sum can be approximated by an integral.

$$\begin{aligned} \underline{\tau}(t) &= -\tilde{\eta} \sum_{p=0}^{\infty} e^{-p} \underline{\dot{\gamma}}(t-pt_0) \quad t'_p \equiv t-pt_0 \\ &= -\tilde{\eta} \sum_{p=0}^{\infty} e^{-(t-t'_p)/t_0} \underline{\dot{\gamma}}(t'_p) = -\tilde{\eta} \frac{(\text{area})}{(-t_0)} \end{aligned}$$

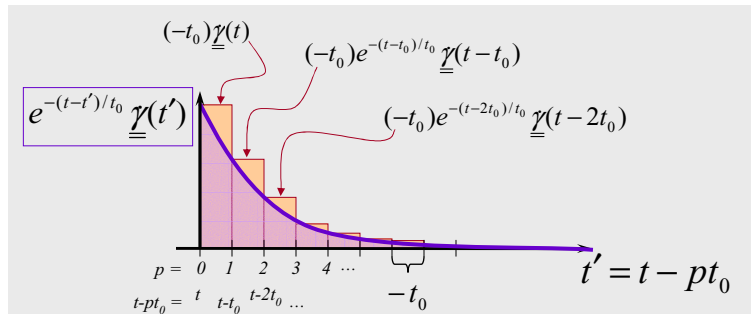


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This sum can be approximated by this integral.

$$\underline{\tau}(t) = -\tilde{\eta} \frac{(\text{area})}{(-t_0)} \approx \frac{\tilde{\eta}}{t_0} \int_t^{-\infty} e^{-(t-t')/t_0} \underline{\gamma}(t') dt'$$

$$\underline{\tau}(t) = - \int_{-\infty}^t \left(\frac{\tilde{\eta}}{t_0} \right) e^{-(t-t')/t_0} \underline{\gamma}(t') dt'$$



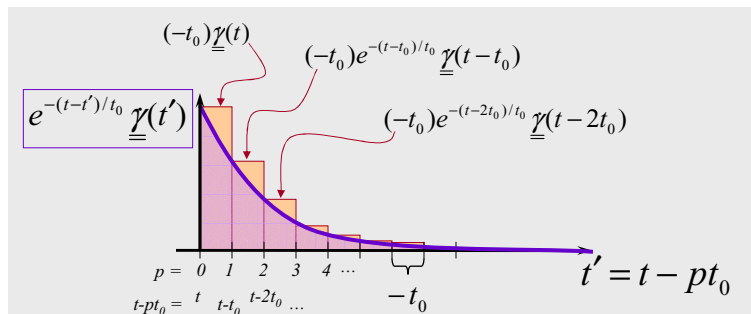
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$$\underline{\tau}(t) = - \int_{-\infty}^t \left(\frac{\tilde{\eta}}{t_0} \right) e^{-(t-t')/t_0} \underline{\gamma}(t') dt'$$

(Note: this is an underestimate of our previous decaying function, but the choice of function is arbitrary)



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$$\underline{\underline{\tau}}(t) = - \int_{-\infty}^t \left(\frac{\tilde{\eta}}{t_0} \right) e^{-(t-t')/t_0} \underline{\underline{\gamma}}(t') dt'$$

**Maxwell Model
(integral
version)**

$$\underline{\underline{\tau}}(t) = - \int_{-\infty}^t \left(\frac{\eta_0}{\lambda} \right) e^{-(t-t')/\lambda} \underline{\underline{\gamma}}(t') dt'$$

Two parameters:

Zero-shear viscosity η_0 – gives the value of the steady shear viscosity

Relaxation time λ - quantifies how fast memory fades