Chapter 3: Newtonian Fluids

Navier-Stokes Equation

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

• Derive governing equations (mass and momentum balances)
• Solve governing equations for velocity and stress fields

QUICK START

First, before we get deep into derivation, let’s do a Navier-Stokes problem to get you started in the mechanics of this type of problem solving.
EXAMPLE: Drag flow between infinite parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- Very wide, long
- Uniform pressure

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} \]

EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- Infinitely wide, long

\[ \begin{align*}
  x_1 &= 0 \\
  p &= P_o \\
  x_1 &= L \\
  p &= P_L
\end{align*} \]
We got a Quick Start with Newtonian problem solving...

Now...
Back to exploring the origin of the equations (so we can adapt to non-Newtonian)

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**Chapter 3: Newtonian Fluid Mechanics**

**TWO GOALS**

- Derive governing equations (mass and momentum balances)
- Solve governing equations for velocity and stress fields

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**Mass Balance**

Consider an arbitrary control volume \( V \) enclosed by a surface \( S \)

\[
\begin{align*}
\text{rate of increase} & = \text{net flux of} \\
\text{of mass in } CV & \text{ mass into } CV
\end{align*}
\]
Consider an arbitrary volume $V$ enclosed by a surface $S$.

The mass balance equation can be expressed as:

\[
\left( \text{rate of increase of mass in } V \right) = \frac{d}{dt} \left( \iiint_V \rho \, dV \right)
\]

And the net flux of mass into $V$ through surface $S$ is:

\[
\left( \text{net flux of mass into } V \text{ through surface } S \right) = -\iint_S \rho \hat{n} \cdot \mathbf{v} \, dS
\]
Chapter 3: Newtonian Fluid Mechanics

Mass Balance (continued)

\[
\frac{d}{dt} \left( \iiint_V \rho \, dV \right) = -\iiint_S \rho \, \hat{n} \cdot \vec{v} \, dS
\]

\[
\iiint_V \frac{\partial \rho}{\partial t} \, dV = -\iiint_S \hat{n} \cdot (\rho \vec{v}) \, dS
\]

\[
= -\iiint_V \nabla \cdot (\rho \vec{v}) \, dV
\]

\[
\iiint_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right) \, dV = 0
\]

Since \( V \) is arbitrary,

Continuity equation: microscopic mass balance

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]
Chapter 3: Newtonian Fluid Mechanics

Polymer Rheology

Mass Balance (continued)

Continuity equation (general fluids)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]
\[ \frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \rho = 0 \]
\[ \frac{D \rho}{Dt} + \rho (\nabla \cdot \mathbf{v}) = 0 \]

For \( \rho = \) constant (incompressible fluids):

\[ \nabla \cdot \mathbf{v} = 0 \]

Momentum Balance

Consider an arbitrary control volume \( V \) enclosed by a surface \( S \)

Momentum is conserved.

\[
\left( \text{rate of increase of momentum in } CV \right) = \left( \text{net flux of momentum into } CV \right) + \left( \text{sum of forces on } CV \right)
\]

- resembles the rate term in the mass balance
- resembles the flux term in the mass balance
- Forces: body (gravity) molecular forces
Momentum Balance

\[ \frac{\text{rate of increase of momentum in } V}{V} = \frac{d}{dt} \left( \iiint_V \rho \, dv \right) \]
\[ = \iiint_V \frac{\partial}{\partial t} (\rho v) \, dv \]

Leibniz rule

\[ \text{net flux of momentum into } V = \iiint_S \hat{n} \cdot (\rho v) \, dS \]
\[ = -\iiint_V \nabla \cdot (\rho v) \, dv \]

Gauss Divergence Theorem
Forces on $V$

Body Forces (non-contact)

\[
\left( \text{force on } V \right)_{\text{due to } g} = \iiint_V \rho g \, dV
\]

Molecular Forces (contact) – this is the tough one

\[
\vec{f} = \left[ \text{stress at } P \right] \frac{dS}{dS}
\]

We need an expression for the state of stress at an arbitrary point $P$ in a flow.
Think back to the molecular picture from chemistry:

The specifics of these forces, connections, and interactions must be captured by the molecular forces term that we seek.

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• We will concentrate on expressing the molecular forces mathematically;
• We leave to later the task of relating the resulting mathematical expression to experimental observations.

First, choose a surface:
• arbitrary shape
• small

\[
\text{stress} \at \text{at } P \on \text{dS} = f \text{ dS}
\]

What is \( f \)?
Consider the forces on three mutually perpendicular surfaces through point $P$:

We can write these vectors in a Cartesian coordinate system:

$$a = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 = \Pi_1 \hat{e}_1 + \Pi_{12} \hat{e}_2 + \Pi_{13} \hat{e}_3$$

**Molecular Forces** (continued)

- $a$ is stress on a “1” surface at $P$
- $b$ is stress on a “2” surface at $P$
- $c$ is stress on a “3” surface at $P$

stress on a “1” surface in the 1-direction
**Molecular Forces** (continued)

\[ a = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \]
\[ = \Pi_{11} \hat{e}_1 + \Pi_{12} \hat{e}_2 + \Pi_{13} \hat{e}_3 \]
\[ b = b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3 \]
\[ = \Pi_{21} \hat{e}_1 + \Pi_{22} \hat{e}_2 + \Pi_{23} \hat{e}_3 \]
\[ c = c_1 \hat{e}_1 + c_2 \hat{e}_2 + c_3 \hat{e}_3 \]
\[ = \Pi_{31} \hat{e}_1 + \Pi_{32} \hat{e}_2 + \Pi_{33} \hat{e}_3 \]

So far, this is nomenclature; next we relate these expressions to force on an arbitrary surface.

How can we write \( f \) (the force on an arbitrary surface \( dS \)) in terms of the \( \Pi_{pk} \)?

\[ f = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3 \]

- \( f_1 \) is force on \( dS \) in 1-direction
- \( f_2 \) is force on \( dS \) in 2-direction
- \( f_3 \) is force on \( dS \) in 3-direction

There are three \( \Pi_{pk} \) that relate to forces in the 1-direction:

\[ \Pi_{11}, \Pi_{21}, \Pi_{31} \]
How can we write \( f \) (the force on an arbitrary surface \( dS \)) in terms of the quantities \( P_{pk} \)?

\[
\text{first part: } \left\{ \begin{array}{c}
\hat{n} \cdot \hat{e}_i dS \\
\left( \Pi_{11} \right) \left\{ \text{projection of } \right.
\left. dA \text{ onto the } \right.
\left. 1\text{-surface} \right) = \Pi_{11} \hat{n} \cdot \hat{e}_i dS \\
\left( \text{force} \right) \cdot \left( \text{area} \right)
\end{array} \right.
\]

\( f_1 \), the force on \( dS \) in 1-direction, can be broken into three parts associated with the three stress components:

\[
\Pi_{11}, \Pi_{21}, \Pi_{31}
\]

\( f_1 \), the force on \( dS \) in 1-direction, is composed of three parts:

\[
\text{first part: } \left\{ \begin{array}{c}
\left( \Pi_{11} \right) \left\{ \text{projection of } \right.
\left. dA \text{ onto the } \right.
\left. 1\text{-surface} \right) = \Pi_{11} \hat{n} \cdot \hat{e}_i dS \\
\left( \Pi_{21} \right) \left\{ \text{projection of } \right.
\left. dA \text{ onto the } \right.
\left. 2\text{-surface} \right) = \Pi_{21} \hat{n} \cdot \hat{e}_2 dS \\
\left( \Pi_{31} \right) \left\{ \text{projection of } \right.
\left. dA \text{ onto the } \right.
\left. 3\text{-surface} \right) = \Pi_{31} \hat{n} \cdot \hat{e}_3 dS
\end{array} \right.
\]

the sum of these three = \( f_1 \)
Molecular Forces (continued)

\[ f_i = \Pi_{1i} \hat{n} \cdot \hat{e}_1 \, dS + \Pi_{2i} \hat{n} \cdot \hat{e}_2 \, dS + \Pi_{3i} \hat{n} \cdot \hat{e}_3 \, dS \]

Using the distributive law:

\[ f_i = \hat{n} \cdot (\Pi_{1i} \hat{e}_1 + \Pi_{2i} \hat{e}_2 + \Pi_{3i} \hat{e}_3) \, dS \]
The same logic applies in the 2-direction and the 3-direction

\[
\begin{align*}
\mathbf{f}_1 &= \hat{n} \cdot (\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3) \, dS \\
\mathbf{f}_2 &= \hat{n} \cdot (\Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3) \, dS \\
\mathbf{f}_3 &= \hat{n} \cdot (\Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3) \, dS
\end{align*}
\]

Assembling the force vector:

\[
\mathbf{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3 \\
= dS \hat{n} \cdot \left( (\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3) \hat{e}_1 \\
+ dS \hat{n} \cdot (\Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3) \hat{e}_2 \\
+ dS \hat{n} \cdot (\Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3) \hat{e}_3 \right)
\]

Molecular Forces (continued)

Assembling the force vector:

\[
\mathbf{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3 \\
= dS \hat{n} \cdot \left( (\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3) \hat{e}_1 \\
+ dS \hat{n} \cdot (\Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3) \hat{e}_2 \\
+ dS \hat{n} \cdot (\Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3) \hat{e}_3 \right)
\]

linear combination of dyadic products = tensor
Assembling the force vector:

\[
\mathbf{f} = dS \mathbf{n} \cdot \left[ \Pi_{11} \mathbf{e}_1 \mathbf{e}_1 + \Pi_{21} \mathbf{e}_2 \mathbf{e}_1 + \Pi_{31} \mathbf{e}_3 \mathbf{e}_1 \\
+ \Pi_{12} \mathbf{e}_1 \mathbf{e}_2 + \Pi_{22} \mathbf{e}_2 \mathbf{e}_2 + \Pi_{32} \mathbf{e}_3 \mathbf{e}_2 \\
+ \Pi_{13} \mathbf{e}_1 \mathbf{e}_3 + \Pi_{23} \mathbf{e}_2 \mathbf{e}_3 + \Pi_{33} \mathbf{e}_3 \mathbf{e}_3 \right]
\]

\[
= dS \mathbf{n} \cdot \sum_{p=1}^{3} \sum_{m=1}^{3} \Pi_{pm} \mathbf{e}_p \mathbf{e}_m
\]

\[
= dS \mathbf{n} \cdot \Pi_{pm} \mathbf{e}_p \mathbf{e}_m
\]

Total stress tensor (molecular stresses)

Momentum Balance (continued)

\[
\begin{aligned}
\left( \text{rate of increase of momentum in } V \right) &= \left( \text{net flux of momentum into } V \right) + \left( \text{sum of forces on } V \right) \\
\iiint_V \frac{\partial}{\partial t} (\rho \mathbf{v}) dV &= -\iiint_V \nabla \cdot (\rho \mathbf{v}) dV + \iiint_V \rho g dV + \text{molecular forces}
\end{aligned}
\]

We use a stress sign convention that requires a negative sign here.

\[
\text{molecular forces} = \iiint_S \left( \text{molecular forces on} \right) dS
\]

\[
= \iiint_S \hat{n} \cdot (-\Pi) dS
\]

\[
= \iiint_V \nabla \cdot (-\Pi) dV
\]

\[\text{Gauss Divergence Theorem}\]
Momentum Balance (continued)

\[ \left( \text{rate of increase of momentum in } V \right) = \left( \text{net flux of momentum into } V \right) + \left( \text{sum of forces on } V \right) \]

\[ \iiint_V \frac{\partial}{\partial t} (\rho\mathbf{v}) \, dV = -\iiint_V \nabla \cdot (\rho \mathbf{v}) \, dV + \iiint_V \rho g \, dV + \text{molecular forces} \]

**UR/Bird choice:**
positive compression (pressure is positive)

**Gauss Divergence Theorem**

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Momentum Balance (continued)

\[ \mathbf{F}_{\text{on surface}} = \iint_S \hat{n} \cdot (-\mathbf{\Pi}) \, dS = \iint_S \hat{n} \cdot (\mathbf{\Pi}) \, dS \]

\[ \mathbf{\Pi}_{yx} \quad \mathbf{\Pi}_{yx} \quad \mathbf{\Pi}_{yx} \quad \mathbf{\Pi}_{yx} \]

**UR/Bird choice:**
fluid at lesser \( y \) exerts force on fluid at greater \( y \)

**IFM/Mechanics choice:** (opposite)

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Momentum Balance (continued)

Final Assembly:

\[
\begin{align*}
\left(\text{rate of increase of momentum in } V\right) &= \left(\text{net flux of momentum into } V\right) + \left(\text{sum of forces on } V\right) \\
\iiint_V \frac{\partial}{\partial t} (\rho v) \, dV &= -\iiint_V \nabla \cdot (\rho v v) \, dV + \iiint_V \rho g \, dV - \iiint_V \nabla \cdot \Pi \, dV \\
\iiint_V \left[ \frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v v) - \rho g + \nabla \cdot \Pi \right] \, dV &= 0
\end{align*}
\]

Because \( V \) is arbitrary, we may conclude:

\[
\frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v v) - \rho g + \nabla \cdot \Pi = 0
\]

Microscopic momentum balance

After some rearrangement:

\[
\begin{align*}
\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) &= -\nabla \cdot \Pi + \rho g \\
\rho \frac{Dv}{Dt} &= -\nabla \cdot \Pi + \rho g
\end{align*}
\]

Equation of Motion

Now, what to do with \( \Pi \)?
Momentum Balance  (continued)  Polymer Rheology

Now, what to do with \( \Pi \)?  Pressure is part of it.

Pressure

definition: An isotropic force/area of molecular origin. Pressure is the same on any surface drawn through a point and acts normally to the chosen surface.

\[
\text{pressure} = p I = p \hat{e}_1 \hat{e}_1 + p \hat{e}_2 \hat{e}_2 + p \hat{e}_3 \hat{e}_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}_{123}
\]

Test: what is the force on a surface with unit normal \( \hat{n} \)?

Extra Molecular Stresses

definition: The extra stresses are the molecular stresses that are not isotropic

\[
\tau = \Pi - p I
\]

Extra stress tensor, i.e. everything complicated in molecular deformation

Now, what to do with \( \tau \)?  This becomes the central question of rheological study
Momentum Balance (continued)

Stress sign convention affects any expressions with
or
\[ \Pi, \tilde{\Pi} \Rightarrow \tau, \tilde{\tau} \]
\[ \Pi \equiv \tau + p \mathbf{I} \]
\[ \tilde{\Pi} \equiv \tilde{\tau} - p \mathbf{I} \]

UR/Bird choice: fluid at lesser y exerts force on fluid at greater y
(IFM/Mechanics choice: opposite)

Constitutive equations for Stress

- are tensor equations
- relate the velocity field to the stresses generated by molecular forces
- are based on observations (empirical) or are based on molecular models (theoretical)
- are typically found by trial-and-error
- are justified by how well they work for a system of interest
- are observed to be symmetric

\[ \tau = f(\nabla \psi, \text{material properties}) \]

Observation: the stress tensor is symmetric
Momentum Balance  (continued)  Polymer Rheology

Microscopic momentum balance

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \rho \mathbf{g} \]

Equation of Motion

In terms of the extra stress tensor:

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \rho \mathbf{g} \]

Equation of Motion

Cauchy Momentum Equation

Components in three coordinate systems (our sign convention):


Newtonian Constitutive equation

\[ \mathbf{T} = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

- for incompressible fluids (see text for compressible fluids)
- is empirical
- may be justified for some systems with molecular modeling calculations

Note: \[ \mathbf{T} = +\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \] (IFM choice: opposite)
How is the Newtonian Constitutive equation related to Newton's Law of Viscosity?

\[ \tau = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

\[ \tau_{21} = -\mu \frac{\partial \mathbf{v}_1}{\partial x_2} \]

• incompressible fluids
• rectilinear flow (straight lines)
• no variation in \( x_3 \)-direction

Back to the momentum balance . . .

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \tau + \rho g \]

\[ \tau = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

We can incorporate the Newtonian constitutive equation into the momentum balance to obtain a momentum-balance equation that is specific to incompressible, Newtonian fluids.
Navier-Stokes Equation

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

- incompressible fluids
- Newtonian fluids

Note: The Navier-Stokes is unaffected by the stress sign convention.

Next?

Navier-Stokes Equation

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

Newtonian Problem Solving
EXAMPLE: Drag flow between infinite parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- Very wide, long
- Uniform pressure

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}
\]

EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- Infinitely wide, long
EXAMPLE: Poiseuille flow in a tube

- Newtonian
- Steady state
- Incompressible fluid
- Long tube

EXAMPLE: Torsional flow between parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- \( v_\theta = zf(r) \)
Done with Newtonian Fluids.

Let’s move on to Standard Flows

Chapter 4: Standard Flows

Newtonian fluids: \( \tau = -\mu \frac{\partial v}{\partial y} \)

non-Newtonian fluids: \( \tau \neq -\mu \frac{\partial v}{\partial y} \)

How can we investigate non-Newtonian behavior?