What are material functions, and why do we need them?
What are material functions and why do we need them?

Constitutive equation

\[ \tau = -\varepsilon = -\mu(\nabla v + (\nabla v)^T) \]

Momentum balance

\[ \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p - \varepsilon + \rho g \]

Flow scenarios

\[ \rho, p(x, y, z) \]

When we know \( \tau(v) \), the flow modeling effort is expended on developing the flow scenario and solving the math.

What are material functions and why do we need them?

Constitutive equation

\[ \tau = \varepsilon = \tau(v) \]

Momentum balance

\[ \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p - \varepsilon + \rho g \]

Flow scenarios

\[ \rho, p(x, y, z) \]

When we don't know \( \tau(v) \), this approach is a nonstarter.
What are material functions and why do we need them?

**Hypothetical Constitutive equation**

\[ \mathbf{f} = -\mathbf{f} = \mathbf{f}(\psi) \]

**Standard Velocity and pressure fields**

\[ \psi = \begin{pmatrix} \zeta(t) \times 2 \\ 0 \\ 0 \end{pmatrix} \] \( \mathbf{p} = p_0 \)

**Material Functions (stress responses)**

\[ \eta(\psi) = \frac{\tau_{21}}{y_0} = \frac{-\tau_{21}}{y_0} \]

\[ \Psi_1(\psi) = \frac{\tau_{11} - \tau_{22}}{y_0} = \frac{-(\tau_{11} - \tau_{22})}{y_0} \]

\[ \Psi_2(\psi) = \frac{\tau_{22} - \tau_{33}}{y_0} = \frac{-(\tau_{22} - \tau_{33})}{y_0} \]

**New approach**

**Ability to measure stresses**

**Material Functions**

Meet here:

**Material Functions**

**Measuring**

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What are material functions and why do we need them?

**Modeling**

The velocity field is fixed (standard flows)

- Approach stress/deformation investigations from two directions (modeling, measuring) to reveal the physics;

- Material functions organize comparisons

**Measuring**

Material Functions

- Ability to measure stresses

Meet here: Material Functions

- Ability to measure stresses

Material Functions (stress responses)

\[ \varepsilon = \begin{pmatrix} \varepsilon_{xx} & 0 & \varepsilon_{xy} \\ 0 & \varepsilon_{yy} & 0 \\ \varepsilon_{xy} & 0 & \varepsilon_{zz} \end{pmatrix} \]

\[ \sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \]

Standard Velocity and pressure fields

\[ \varphi = \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} \]

\[ \psi = \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix} \]

\[ p = p_0 \]

Material Functions

\[ \varepsilon = \frac{1}{2} \left( \nabla \varphi + (\nabla \varphi)^T \right) \]

\[ \sigma = 2 \mu \varepsilon + \lambda \nabla \cdot \varphi \]

\[ p = \frac{1}{3} \lambda \nabla \cdot \varphi \]

Meet here:

- Material Functions

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What are material functions and why do we need them?

For non-Newtonian fluids:

- We do not know the stress/deformation relationship \( \mathbf{\tau}(\mathbf{V}) \).
- We approach stress/deformation investigations from two directions (modeling, measuring) to reveal the physics;
- Material functions organize comparisons

Investigating Stress/Deformation Relationships (Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( \mathbf{\tau}(\mathbf{V}) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

Let’s get started:

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 Investing Stress/Deformation Relationships (Rheology)

1) Choose a material function

- 1. Choice of flow (shear or elongation)
- 2. Choice of time dependence of $\dot{\gamma}$ or $\dot{\varepsilon}$
- 3. Material functions definitions: will be based on $\tau_{12}, N_1, N_2$ in shear or $\tau_{22} - \tau_{11}, \tau_{22} - \tau_{11}$ in elongational flows.

Steady Shear Flow Material Functions

- Imposed Kinematics:
  - $\dot{\gamma}(t) = \dot{\gamma}_0 = \text{constant}$

- Material Stress Response:
  - $\tau_{21}(t)$
  - $N_1(t)$

- Material Functions:
  - Viscosity $\eta(\dot{\gamma}_0) \equiv \frac{\tau_{21}}{\dot{\gamma}_0} = -\frac{\tau_{21}}{\dot{\gamma}_0}$
  - First normal-stress coefficient $\Psi_1(\dot{\gamma}_0) \equiv \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0} = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0}$
  - Second normal-stress coefficient $\Psi_2(\dot{\gamma}_0) \equiv \frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0} = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0}$
How do we predict material functions?

**New Approach**

**Constitutive equation**

\[
\mathbf{t} = -\mathbf{\dot{e}} = \mathbf{f}(\mathbf{v})
\]

**Standard Velocity and pressure fields**

\[
\mathbf{v} = \begin{pmatrix}
\dot{\gamma}(t)x_2 \\
0 \\
0
\end{pmatrix}
\]

\[
p = p_0
\]

**Material Functions (stress responses)**

\[
\eta(\dot{\gamma}_0) = \frac{\tau_{21}}{\dot{\gamma}_0} = \frac{-\tau_{21}}{\dot{\gamma}_0}
\]

\[
\Psi_1(\dot{\gamma}_0) = \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0} = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0}
\]

\[
\Psi_2(\dot{\gamma}_0) = \frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0} = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0}
\]

**How do we predict material functions?**

*We must know the Constitutive Equation.*
What does the Newtonian Fluid model predict in steady shearing?

\[
\tau = -\mu \dot{\gamma} = -\mu \left[ \nabla \nu + (\nabla \nu)^T \right]
\]

2) Predict what Newtonian fluids would do

You try.
What do we measure for these material functions?

(for polymer solutions, for example)

3) See what non-Newtonian fluids do

Steady shear viscosity and first normal stress coefficient

Figure 6.1, p. 170 Menzes and Graessley conc. PB solution
3) See what non-Newtonian fluids do

Steady shear viscosity and first normal stress coefficient

\[ \eta \text{, Poise} \]

\[ \Psi_1 \text{, dynes/cm}^2 \]

\[ \dot{\gamma} \text{, s}^{-1} \]

Figure 6.2, p. 171 Menzes and Graessley conc. PB solution; \( c=0.0676 \text{ g/cm}^3 \)

Figure 6.3, p. 172 Piau et al., linear and branched PDMS

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3) See what non-Newtonian fluids do

Steady shear viscosity and first and second normal stress coefficient

![Graph showing steady shear viscosity and normal stress coefficients](image)

Figure 6.6, p. 174 Magda et al.; Polystyrene solutions

Investigating Stress/Deformation Relationships (Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a $\tau(\dot{\gamma})$
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

4) Hypothesize a $\tau(\nu)$

(how do we do that, exactly?)
What have the investigations of the steady shear material functions taught us so far?

- Newtonian constitutive equation is inadequate
  1. Predicts constant shear viscosity (does not predict rate dependence)
  2. Predicts no shear normal stresses (a nonlinear effect; these stresses are generated for many fluids)
- Behavior depends on the material (chemical structure, molecular weight, concentration)

Can we fix the Newtonian Constitutive Equation?

\[
\tau = -\mu \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right]
\]

Let's replace \( \mu \) with a function of shear rate because we want to predict a rate dependence.

New hypothesis for \( \tau(\mathbf{v}) \):
What are material functions and why do we need them?

Constitutive equation: \( \dot{\tau} = -\dot{\gamma} = \tau(\gamma) \)

Standard Velocity and pressure fields:

\[ v = \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}, \quad p = p_0 \]

Material Functions (stress responses):

\[ \eta(\dot{\gamma}_0) \equiv \frac{\dot{\tau}_{21}}{\dot{\gamma}_0} = \frac{-\tau_{21}}{\dot{\gamma}_0} \]

\[ \psi_1(\dot{\gamma}_0) \equiv \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0^2} = -\frac{r_{11} - r_{22}}{\dot{\gamma}_0^2} \]

\[ \psi_2(\dot{\gamma}_0) \equiv \frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0^2} = -\frac{(r_{22} - r_{33})}{\dot{\gamma}_0^2} \]

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Investigating Stress/Deformation Relationships (Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( f(\gamma) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

5) **Predict the material function** (with new \( f(\gamma) \))

What does this model predict for steady shear viscosity?

\[
\tau = -M(\gamma_0)\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right]
\]

---

You try.
5) Predict the material function

What does this model predict for steady shear viscosity?

\[ \tau = -M(\dot{\gamma}_0) \left[ \nabla \dot{\gamma} + (\nabla \dot{\gamma})^T \right] \]

Answer:

\[ \eta = M(\dot{\gamma}_0) \]

5) Predict the material function

If we choose:

\[ M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m \dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases} \]

Fake-O Model®

If we choose: (we are forcing the observed rate dependence)

\[ M(\dot{\gamma}_0) \]

\[ \text{slope} = (n-1) \]

Problem solved!
The model and the experiments for \( \eta(\dot{\gamma}) \) match.
5) Predict the material function

But what about the normal stresses?

\[
\tau = -M(\gamma_0)\left[\nabla \gamma + (\nabla \gamma)^T\right]
\]

\[
\nabla \gamma = \begin{pmatrix}
0 & 0 & 0 \\
\gamma_0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}_{123}
\]

\[
\dot{\gamma} = \begin{pmatrix}
0 & \gamma_0 & 0 \\
\gamma_0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}_{123}
\]

\[
M(\gamma_0) = \begin{cases}
M_0 & \gamma_0 < \gamma_c \\
M_0 \gamma_0^{n-1} & \gamma_0 \geq \gamma_c
\end{cases}
\]

\[
\tau = \begin{pmatrix}
0 & -M(\gamma_0)\gamma_0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}_{123}
\]

\[\Rightarrow \Psi_1 = \Psi_2 = 0\]

6) Compare with what non-Newtonian fluids do

Investigating Stress/Deformation Relationships (Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \(f(\tau)\)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.
6) Compare with what non-Newtonian fluids do

Menzes and Graessley, conc. PB solution; 350 kg/mol

Fake-O Model Predictions:

Steady shear:

\[ \eta = \begin{cases} M_0 & \dot{\gamma} \leq \dot{\gamma}_c \\ \frac{1}{m} \dot{\gamma}_0^{n-1} & \dot{\gamma}_0 > \dot{\gamma}_c \end{cases} \]

\[ \Psi_1 = 0 \]

\[ \Psi_2 = 0 \]

Polymer Behavior:

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6) Compare with what non-Newtonian fluids do

Fake-O Model Predictions:

Steady shear:
\[ \eta = \begin{cases} M_0 & \dot{\gamma}_0 \leq \dot{\gamma}_c \\ m \dot{\gamma}_0^{n-1} & \dot{\gamma}_0 > \dot{\gamma}_c \end{cases} \]

\[ \Psi_1 = 0 \]
\[ \Psi_2 = 0 \]

Polymer Behavior:

Why did we get 0 for the normal stresses?

\[ \tau = -M(\dot{\gamma}_0)\left[\nabla \dot{\gamma} + (\nabla \dot{\gamma})^T\right] \]

\[ M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m \dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases} \]

\[ \tau = -M(\dot{\gamma}_0) \begin{pmatrix} 0 & \dot{\gamma}_0 & 0 \\ \dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \Rightarrow \Psi_1 = \Psi_2 = 0 \]

Need to try something else . . .

\[ \tau = -\mu \dot{\gamma} + f(\dot{\gamma}) \]
\[ \tau = f(\dot{\gamma}) \nabla \dot{\gamma} \cdot (\nabla \dot{\gamma})^T \]
\[ \tau = A[\nabla \dot{\gamma} \cdot (\nabla \dot{\gamma})^T] + B \nabla \dot{\gamma} + C(\nabla \dot{\gamma})^T \]

It appears that \( \tau \) should not be simply proportional to \( \dot{\gamma} \).
Investigating **Stress/Deformation Relationships**  

(Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( f(\gamma) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

7) Reflect, learn, revise model, repeat

What are material functions and why do we need them?

- We do not know the stress/deformation relationship \( (\mu|\gamma) \)
- We approach stress/deformation investigations from two directions (modeling, measuring) to reveal the physics
- Material functions organize comparisons

For non-Newtonian fluids:

1. Reflect
2. Learn
3. Revise model
4. Propose new \( f(\gamma) \)
5. Repeat cycle….

---

What shall we guess next?

To sort out how to fix the Newtonian equation, we need more observations (to give us ideas).

Let’s try another material function that’s not a steady flow (but stick to shear).

1) **Choose a material function**
Investigating **Stress/Deformation** Relationships  (Rheology)

1.  **Choose a material function**

   1.  Choice of flow (shear or elongation)
   
   \[
   \mathbf{v} = \begin{pmatrix} \zeta(t)x_2 \\ 0 \\ 0 \end{pmatrix}
   \]

   2.  Choice of time dependence of \( \zeta(t) \) or \( \dot{\zeta}(t) \)

   3.  Material functions definitions: will be based on \( \tau_{21}, N_1, N_2 \) in shear or \( \tau_{22} - \tau_{11}, \tau_{22} - \tau_{11} \) in elongational flows.

---

**Start-up of Steady** Shear Flow Material Functions

**Imposed Kinematics:**

\[
\mathbf{v} = \begin{pmatrix} \zeta(t)x_2 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\zeta(t) = \begin{cases} 
0 & t < 0 \\
\dot{y}_0 & t \geq 0
\end{cases}
\]

**Material Stress Response:**

Material Functions:

Shear stress growth function

\[
\eta^+(t, \dot{y}_0) \equiv \frac{\tau_{21}(t)}{\dot{y}_0} = \frac{-\tau_{21}(t)}{\dot{y}_0}
\]

First normal-stress growth coefficient

\[
\Psi_1^+(t, \dot{y}_0) \equiv \frac{\tau_{11} - \tau_{22}}{\dot{y}_0}
\]

Second normal-stress growth coefficient

\[
\Psi_2^+(t, \dot{y}_0) \equiv \frac{\tau_{22} - \tau_{33}}{\dot{y}_0}
\]
What does the Newtonian Fluid model predict in start-up of steady shearing?

\[ \mathbf{T} = - \mu \dot{\mathbf{\gamma}} = - \mu \left[ \nabla \mathbf{\gamma} + (\nabla \mathbf{\gamma})^T \right] \]

Again, since we know \( \mathbf{\gamma} \), we can just substitute it into the constitutive equation and calculate the stresses.

2) Predict what Newtonian fluids would do (round 2)

What does the Newtonian Fluid constitutive equation predict in start-up of steady shearing?

\[ \mathbf{T} = - \mu \dot{\mathbf{\gamma}} = - \mu \left[ \nabla \mathbf{\gamma} + (\nabla \mathbf{\gamma})^T \right] \]

You try.
2) Predict what Newtonian fluids would do (round 2)

Material functions predicted for *start-up of steady shearing* of a Newtonian fluid

\[
\eta^+(t) = \begin{cases} 
0 & t < 0 \\
\mu & t \geq 0 
\end{cases}
\]

\[
\Psi_1^+ = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} = 0
\]

\[
\Psi_2^+ = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} = 0
\]

Do these predictions match observations?

---

3) See what non-Newtonian fluids do

**What do we measure for these material functions?**

(for polymer solutions, for example)
3) See what non-Newtonian fluids do (round 2)

**Startup of Steady Shearing**

\[ \dot{\gamma}(t) = \begin{cases} 
\dot{\gamma}_1 & t < 0 \\
\dot{\gamma}_0 & t \geq 0 
\end{cases} \]

\[ \eta^* = \frac{-\tau(t)}{\dot{\gamma}_0} \]

Figures 6.49, 6.50, p. 208 Menezes and Graessley, Polybutadiene solution

**Investigating Stress/Deformation Relationships (Rheology)**

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( \dot{\gamma}(t) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

3) See what non-Newtonian fluids do

**What about other non-steady flows?**
Cessation of Steady Shear Flow Material Functions

Imposed Kinematics:
\[
\dot{y} \equiv \begin{pmatrix}
\dot{\gamma}(t)x_2 \\
0 \\
0
\end{pmatrix}_{123}
\]
\[
\dot{\gamma}(t) = \begin{cases}
\dot{\gamma}_0 & t < 0 \\
0 & t \geq 0
\end{cases}
\]
\[
\gamma_{21}(0, t)
\]

Material Stress Response:

Material Functions:

Shear stress decay function
\[
\eta^{-1}(t, \dot{\gamma}_0) \equiv -\frac{\tau_{21}(t)}{\dot{\gamma}_0} = \frac{-\tau_{21}(t)}{\dot{\gamma}_0}
\]

First normal-stress decay coefficient
\[
\Psi_1^{-}(t, \dot{\gamma}_0) \equiv \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0^2}
\]

Second normal-stress decay coefficient
\[
\Psi_2^{-}(t, \dot{\gamma}_0) \equiv \frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0^2}
\]

3) See what non-Newtonian fluids do (round 3)

Cessation of Steady Shearing

\[
\dot{y} \equiv \begin{pmatrix}
\dot{\gamma}(t)x_2 \\
0 \\
0
\end{pmatrix}_{123}
\]
\[
\dot{\gamma}(t) = \begin{cases}
\dot{\gamma}_0 & t < 0 \\
0 & t \geq 0
\end{cases}
\]

Figures 6.51, 6.52, p. 209
Menezes and Graessley, PB soln

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5) Predict the material function

What does the Fake-O model® predict for start-up and cessation of shear?

\[ \xi = -M(\dot{\gamma}_0) \left[ \nabla \dot{\gamma} + (\nabla \dot{\gamma})^T \right] \]

\[ M(\dot{\gamma}_0) = \begin{cases} 
M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\
m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c 
\end{cases} \]

5) Predict the material functions (rounds 2 & 3)

What does the Fake-O model® predict for start-up and cessation of shear?

\[ \xi = -M(\dot{\gamma}_0) \left[ \nabla \dot{\gamma} + (\nabla \dot{\gamma})^T \right] \]

\[ M(\dot{\gamma}_0) = \begin{cases} 
M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\
m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c 
\end{cases} \]

You try.
Investigating Stress/Deformation Relationships (Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a $g(\tau)$
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

6) Compare with what non-Newtonian fluids do

?
### Fake-O Model Material Function Predictions:

**Start-up of steady shear:**

\[
\eta^+(t) = \begin{cases} 
0 & t < 0 \\
M(\dot{\gamma}_0) & t \geq 0 
\end{cases}
\]

where \(M(\dot{\gamma}_0) = \begin{cases} 
M_0 & \dot{\gamma}_0 \leq \dot{\gamma}_c \\
m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 > \dot{\gamma}_c 
\end{cases}\)

\(\Psi_1^+(t) = 0\)

\(\Psi_2^+(t) = 0\)

**Cessation of steady shear:**

\[
\eta^-(t) = \begin{cases} 
0 & t < 0 \\
M(\dot{\gamma}_0) & t \geq 0 
\end{cases}
\]

where \(M(\dot{\gamma}_0) = \begin{cases} 
M_0 & \dot{\gamma}_0 \leq \dot{\gamma}_c \\
m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 > \dot{\gamma}_c 
\end{cases}\)

\(\Psi_1^-(t) = 0\)

\(\Psi_2^-(t) = 0\)

### Observations

- The Fake-O model® predicts an instantaneous stress response, and this is not what is observed for polymers.

- The predicted unsteady material functions depend on the shear rate, which is observed for polymers.

\[
\eta^+ = \eta^+(t, \dot{\gamma}_0)
\]

- No shear normal stresses are predicted.
6) Compare with what non-Newtonian fluids do (rounds 2&3)

\[ \tau = -M(\dot{\gamma}) \left( \mathbf{V} \mathbf{V}^T \right) \]

**Observations**

- The Fake-O model® predicts an instantaneous stress response, and this is not what is observed for polymers
  - Lacks memory
- The predicted unsteady material functions depend on the shear rate, which is observed for polymers
  - Progress here
- No shear normal stresses are predicted
  - Related to nonlinearities

7) Reflect, learn, revise model, repeat

- Reflect
- Learn
- Revise model
- Propose new \( \tau(\dot{\gamma}) \)
- Repeat cycle….
7) Reflect, learn, revise model, repeat

To proceed to better-designed constitutive equations, we need to know more about material behavior, i.e. we need more material functions to predict, and we need measurements of these material functions.

- More non-steady material functions (material functions that tell us about memory)
- Material functions that tell us about nonlinearity (strain)

Investigating Stress/Deformation Relationships (Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( \tau(g) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

1) Choose a material function

\[
\begin{align*}
\mathbf{v} &= \begin{pmatrix} \dot{\xi}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \\
\mathbf{y} &= \begin{pmatrix} \frac{1}{2}\dot{\xi}(t)x_1 \\ -\frac{1}{2}\dot{\xi}(t)x_2 \\ \dot{\xi}(t)x_3 \end{pmatrix}_{123}
\end{align*}
\]

2. Choice of time dependence of \( \xi(t) \) or \( \dot{\xi}(t) \)

3. Material functions definitions: will be based on \( \tau_{21}, N_1, N_2 \) in shear or \( \tau_{22} - \tau_{11}, \tau_{22} - \tau_{11} \) in elongational flows.
1) Choose a material function – Rate based

Summary of shear rate kinematics (part 1)

- **a. Steady**
  - \( \dot{\gamma}(t) \)
  - \( \dot{\gamma}_2(0, t) \)
  - \( \tau_2(t) \)

- **b. Stress Growth**
  - \( \dot{\gamma}(t) \)
  - \( \dot{\gamma}_2(0, t) \)
  - \( \tau_2(t) \)

- **c. Stress Relaxation**
  - \( \dot{\gamma}(t) \)
  - \( \dot{\gamma}_2(0, t) \)
  - \( \tau_2(t) \)

Summary of shear rate kinematics (part 1, rate-based)

- **Strain-rate based**
  - \( \dot{\gamma}(t) \)
  - \( \dot{\gamma}_2(0, t) \)
  - \( \tau_2(t) \)

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The next three families of material functions incorporate the concept of **strain**.

The first three recipe cards were strain-rate based.
The first three recipe cards were strain-rate based.

The second three recipe cards are strain based.

1) Choose a material function – Strain based

Summary of shear rate kinematics *(part 2; strain-based)*
What is strain?

**Strain** is a measure of deformation (change in shape)

We need a way to quantify “change in shape” due to flow. There must be an initial (reference) shape and a final shape (at time of interest).

The problem of change in shape is a difficult, 3-dimensional problem; we can start simple with unidirectional flow (shear).
Steady Shear:

\[
\dot{\psi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

How can we quantify the change in shape?

Define \textbf{shear strain} as relative change in shape:

shear strain = \frac{\Delta u_1}{\Delta x_2} = \begin{pmatrix} \text{Displacement in flow direction} \\ \text{Spacing in gradient direction} \end{pmatrix}

Displacement depends on position.
What is strain? Displacement depends on position

\[ u_1(t_{\text{ref}}, t) = x_1 \text{-displacement of a particle between times } t_{\text{ref}} \text{ and } t \]

\[ \Delta x_2 \]

Fluid particle at time \( t_{\text{ref}} \)

Fluid particle at time \( t \)

\[ x_1(t_{\text{ref}}) \rightarrow x_1(t) \]

\[ u_1 \bigg|_{x_2} = x_1(t) - x_1(t_{\text{ref}}) \]

Steady Shear

\[ \psi \equiv \begin{pmatrix} \gamma_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

At \( x_2 \):}

\[ u_1 \bigg|_{x_2} = x_1(t) - x_1(t_{\text{ref}}) = \gamma_0 x_2 \Delta t \]
What is strain? Displacement depends on position

\[ u_1(t_{ref}, t) = x_1 \] displacement of a particle between times \( t_{ref} \) and \( t \)

\[ x_1(t) = x_1(t_{ref}) + (velocity)(\Delta t) \]
\[ = x_1(t_{ref}) + \dot{y}_0(x_2 + \Delta x_2)(\Delta t) \]

At \( x_2 + \Delta x_2 \):

\[ u_1 \bigg|_{x_2+\Delta x_2} = x_1(t) - x_1(t_{ref}) = \dot{y}_0(x_2 + \Delta x_2)\Delta t \]

Steady Shear

\[ \psi \equiv \begin{pmatrix} \dot{y}_0x_2 \\ 0 \\ 0 \end{pmatrix} \]

Strain in Shear

\[ u_1 \bigg|_{x_2+\Delta x_2} = \dot{y}_0(x_2 + \Delta x_2)(t - t_{ref}) \]

\[ u_1 \bigg|_{x_2} = \dot{y}_0x_2(t - t_{ref}) \]

Shrink \( \Delta x_2 \) to zero:

\[ \lim_{\Delta x_2 \to 0} \frac{u_1|_{x_2+\Delta x_2} - u_1|_{x_2}}{\Delta x_2} = \dot{y}_0(t - t_{ref}) \]

Shear strain:

\[ \gamma_{21}(t_{ref}, t) = \frac{du_1}{dx_2} = \dot{y}_0(t - t_{ref}) \]

(Note that strain is independent of position in this flow)
What is strain?

\[ u_1(t_{\text{ref}}, t) = x_1 \text{-displacement of a particle between times } t_{\text{ref}} \text{ and } t \]

**Steady Shear**

\[ \varepsilon \equiv \begin{pmatrix} \dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

### Strain in Shear

**What about unsteady shear flow?**

\[ \dot{\gamma}_0 (x_2 + \Delta x_2) (t - t_{\text{ref}}) \]

\[ \Delta x_2 = \dot{\gamma}_0 x_2 (t - t_{\text{ref}}) \]

\[ \lim_{\Delta x_2 \to 0} \frac{u_{1x_2 + \Delta x_2} - u_{1x_2}}{\Delta x_2} = \dot{\gamma}_0 (t - t_{\text{ref}}) \]

Shear strain:

\[ \gamma_21(t_{\text{ref}}, t) \equiv \frac{d u_1}{d x_2} = \dot{\gamma}_0 (t - t_{\text{ref}}) \]

(note that strain is independent of position in this flow)

What about *Elongation al flow?*

\[ \dot{\gamma}_0 (x_2 + \Delta x_2) (t - t_{\text{ref}}) \]

\[ \Delta x_2 = \dot{\gamma}_0 x_2 (t - t_{\text{ref}}) \]

\[ \lim_{\Delta x_2 \to 0} \frac{u_{1x_2 + \Delta x_2} - u_{1x_2}}{\Delta x_2} = \dot{\gamma}_0 (t - t_{\text{ref}}) \]

Shear strain:

\[ \gamma_21(t_{\text{ref}}, t) \equiv \frac{d u_1}{d x_2} = \dot{\gamma}_0 (t - t_{\text{ref}}) \]

(note that strain is independent of position in this flow)
What is strain?

More generally:

\[
\text{strain} \equiv \frac{\Delta u_i}{\Delta x_j} = \begin{pmatrix}
\text{Displacement} \\
\text{Spacing}
\end{pmatrix}
\]

\[
y_{ij}(t_{\text{ref}}, t) \equiv \frac{\partial u_i}{\partial x_j}
\]

(this is a tensor)

Deformation (strain)

\[
\tau(t_{\text{ref}}) = \begin{pmatrix}
x_1(t_{\text{ref}}) \\
x_2(t_{\text{ref}}) \\
x_3(t_{\text{ref}})
\end{pmatrix}
\]

\[
\tau(t) = \begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix}
\]

Current position compared to reference position

Relative change in displacement

Current position

Displacement function

Infinite strain

Particle path

Flow

This vector keeps track of the location of a fluid particle as a function of time.

(more on this later)
Can we arrive at the shear strain result a bit more generally?

Shear strain:

\[ \dot{\gamma}_{21}(t_{\text{ref}}, t) = \frac{du_1}{dx_2} = \dot{\gamma}_0(t - t_{\text{ref}}) \]

What is strain? (unsteady shear)

What is the strain in the standard flow steady shear?

Shear strain:

\[ \dot{\gamma}_{21}(t_{\text{ref}}, t) = \frac{du_1}{dx_2} = \dot{\gamma}_0(t - t_{\text{ref}}) \]

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What is the strain in the standard flow steady shear?

\[
\mathbf{y} = \left( \begin{array}{c} \dot{y}_0 x_2 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} \frac{dx_2}{dt} \\ \frac{dx_1}{dt} \\ \frac{dx_3}{dt} \end{array} \right)_{123}
\]

\[
\frac{dx_1}{dt} = \dot{y}_0 x_2
\]

\[
\int_{x_1(t_{ref})}^{x_1(t)} dx_1 = \int_{t_{ref}}^{t} \dot{y}_0 x_2 dt'
\]

\[
x_1(t) - x_1(t_{ref}) = \dot{y}_0 x_2 (t - t_{ref})
\]

We introduce the dummy variable \( \dot{t} \) to keep it distinct from the limit \( t \).
What is strain?

**Deformation in shear flow (strain)**

\[
r(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad r(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}
\]

Initial position

\[
r(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad r(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}
\]

Final position

**Displacement function**

\[
u(t_{ref}, t) \equiv r(t) - r(t_{ref}) = \begin{pmatrix} (t - t_{ref})\dot{y}_0 x_2 \\ 0 \\ 0 \end{pmatrix}
\]

Final-initial

**Shear strain**

\[
\gamma_{21}(t_{ref}, t) = (t - t_{ref})\dot{y}_0
\]

(for steady shear or in unsteady shear for short time intervals)
For unsteady shear, \( \dot{\gamma} \) is a function of time:

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \end{pmatrix} = \begin{pmatrix} \dot{\gamma}(t) x_2 \\ 0 \\ 0 \\ \end{pmatrix}
\]

We can follow the same steps.

\[
\int_{x_1(t_{ref})}^{x_1(t)} dx_1 = \int_{t_{ref}}^{t} \dot{\gamma}(t') x_2 dt'
\]

\[
x_1(t) - x_1(t_{ref}) = x_2 \int_{t_{ref}}^{t} \dot{\gamma}(t') dt'
\]

What is strain?

Unsteady Shear

Deformation in shear flow (strain)

\[
\begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \\ \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \end{pmatrix}
\]

Initial position

\[
\begin{pmatrix} x_1(t_{ref}) + x_2 \int_{t_{ref}}^{t} \dot{\gamma}(t') dt' \\ x_2(t_{ref}) \\ x_3(t_{ref}) \\ \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \end{pmatrix}
\]

final position

Displacement function

\[
u(t_{ref}, t) = \begin{pmatrix} x_1 \int_{t_{ref}}^{t} \dot{\gamma}(t') dt' \\ x_2 \int_{t_{ref}}^{t} \dot{\gamma}(t') dt' \\ 0 \\ \end{pmatrix}
\]

final-initial
What is strain?

**Unsteady Shear**

<table>
<thead>
<tr>
<th><code>y</code></th>
<th><code>0</code></th>
</tr>
</thead>
<tbody>
<tr>
<td><code>0</code></td>
<td><code>123</code></td>
</tr>
</tbody>
</table>

**Displacement function**

\[ u(t_{ref}, t) \equiv \bar{z}(t) - \bar{z}(t_{ref}) = \begin{pmatrix} x_2 \int_{t_{ref}}^{t} \tilde{\zeta}(t')dt' \\ 0 \\ 0 \end{pmatrix}_{123} \]

\[ \gamma_{21}(t_{ref}, t) \equiv \frac{\partial u_1}{\partial x_2} = \frac{du_1}{dx_2} \]

**Shear strain**

\[ \gamma_{21}(t_{ref}, t) = \int_{t_{ref}}^{t} \tilde{\zeta}(t')dt' \quad (\text{for unsteady shear}) \]

---

**Change of Shape**

For shear flow (steady or unsteady):

\[ \gamma_{21}(t_1, t_2) = \int_{t_1}^{t_2} \dot{\gamma}_{21}(t')dt' \]

Note also, by Leibnitz rule:

\[ \frac{d}{dt} \gamma_{21}(t_{ref}, t) = \frac{d}{dt} \int_{t_{ref}}^{t} \dot{\gamma}_{21}(t')dt' = \int_{t_{ref}}^{t} \frac{\partial}{\partial t} \dot{\gamma}_{21}(t')dt' + \dot{\gamma}_{21}(t) \frac{dt}{dt} - \dot{\gamma}_{21}(t_{ref}) \frac{dt_{ref}}{dt} \]

\[ \frac{dy_{21}}{dt} = \dot{\gamma}_{21}(t) \]

Strain at \( t_2 \) with respect to fluid configuration at \( t_1 \) in shear flow (steady or unsteady).

Not a function of the integration limit \( t \)

---

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What is strain? Summary

**Strain** is our measure of deformation (change of shape)

For shear flow (steady or unsteady):

\[ \gamma_{21}(t_1, t_2) = \int_{t_1}^{t_2} \dot{\gamma}_{21}(t') dt' \]

Strain is the integral of strain rate

Strain accumulates as the flow progresses.

\[ \frac{d\gamma_{21}}{dt} = \dot{\gamma}_{21}(t) \]

The time derivative of strain is the strain rate.

The strain rate is the rate of instantaneous shape change.

Now we can continue with material functions based on strain.

---

Practice with strain

- **What is the strain in start-up of steady shear?**
  
  (let \( t_{ref} = -\infty \))

- **What is the strain in cessation of steady shear?**
  
  (let \( t_{ref} = 0 \))
**Start-up of Steady Shear Flow Material Functions**

**Imposed Kinematics:**

\[ \gamma = \begin{pmatrix} \dot{\gamma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

\[ \dot{\gamma}(t) = \begin{cases} \dot{\gamma}_0 & t \leq 0 \\ \tilde{\gamma}_0 & t > 0 \end{cases} \]

\[ \dot{\gamma}(t) \]

\[ \gamma_{21}(0, t) \]

**Material Stress Response:**

\[ \tau_{21}(t) \]

\[ N_1(t) \]

**Material Functions:**

Shear stress growth function

\[ \eta^+(t, \dot{\gamma}_0) \equiv \frac{\tau_{21}(t)}{\dot{\gamma}_0} = \frac{-\tau_{21}(t)}{\dot{\gamma}_0} \]

First normal-stress growth coefficient

\[ \Psi_1^+(t, \dot{\gamma}_0) \equiv \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0} \]

Second normal-stress growth coefficient

\[ \Psi_2^+(t, \dot{\gamma}_0) \equiv \frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0} \]

**Cessation of Steady Shear Flow Material Functions**

**Imposed Kinematics:**

\[ \gamma = \begin{pmatrix} \dot{\gamma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

\[ \dot{\gamma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases} \]

\[ \dot{\gamma}(t) \]

\[ \gamma_{21}(0, t) \]

**Material Stress Response:**

\[ \tau_{21}(t) \]

\[ N_1(t) \]

**Material Functions:**

Shear stress decay function

\[ \eta^-(t, \dot{\gamma}_0) \equiv \frac{\tau_{21}(t)}{\dot{\gamma}_0} = \frac{-\tau_{21}(t)}{\dot{\gamma}_0} \]

First normal-stress decay coefficient

\[ \Psi_1^-(t, \dot{\gamma}_0) \equiv \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0} \]

Second normal-stress decay coefficient

\[ \Psi_2^-(t, \dot{\gamma}_0) \equiv \frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0} \]

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What shall we do next?

To sort out how to model non-Newtonian behavior, we need more observations (to give us ideas).

We want to try another material function.

1) Choose a material function

Terminology:

1. Choice of flow (shear or elongation)

\[ \mathbf{y} = \begin{pmatrix} \zeta(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

2. Choice of time dependence of \( \zeta(t) \) or \( \dot{\zeta}(t) \)

3. Material functions definitions: will be based on \( \tau_{21}, N_1, N_2 \) in shear or \( \tau_{22} - \tau_{11}, \tau_{22} - \tau_{11} \) in elongational flows.
Summary of shear rate kinematics (part 2; strain-based)

1) Choose a material function – Strain based

**Step Strain** Shear Flow Material Functions

**Imposed Kinematics:**
\[
\gamma(t) = \begin{cases} 
\gamma(0) & t \leq 0 \\
\frac{\gamma_0}{\varepsilon} & 0 < t < \varepsilon \\
0 & t \geq \varepsilon 
\end{cases}
\]

\[
\dot{\gamma}(t) = \lim_{\varepsilon \to 0} \frac{\gamma(0)}{\varepsilon} 
\]

**Material Stress Response:**
\[
\tau_{21}(t)
\]

**Material Functions:**
- Relaxation modulus:
  \[
  G(t, \gamma_0) = \frac{\dot{\tau}_{21}(t, \gamma_0)}{\gamma_0} = \frac{-\tau_{21}(t, \gamma_0)}{\gamma_0}
  \]
- First normal-stress relaxation modulus:
  \[
  G_{\psi_1}(t, \gamma_0) = \frac{\dot{\tau}_{11}(t, \gamma_0) - \dot{\tau}_{22}(t, \gamma_0)}{\gamma_0}
  \]
- Second normal-stress relaxation modulus:
  \[
  G_{\psi_2}(t, \gamma_0) = \frac{\dot{\tau}_{22}(t, \gamma_0) - \dot{\tau}_{33}(t, \gamma_0)}{\gamma_0}
  \]
What is the strain in the step strain flow?

\[
\gamma_{21}(-\infty, t) = \int_{-\infty}^{t} \dot{\gamma}_{21}(t') \, dt'
\]

\[
= \int_{-\infty}^{t} \lim_{\varepsilon \to 0} \begin{cases} 
0 & t' < 0 \\
\frac{\gamma_0}{\varepsilon} & 0 \leq t' < \varepsilon \\
0 & t \geq \varepsilon
\end{cases} \, dt'
\]

\[
= \lim_{\varepsilon \to 0} \int_{0}^{\varepsilon} \frac{\gamma_0}{\varepsilon} \, dt'
\]

\[
= \gamma_0 \quad \text{The strain imposed is a constant}
\]
3) See what non-Newtonian fluids do

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a g(t)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

Step shear strain - strain dependence

Figure 6.57, p. 212 Einaga et al.; PS soln

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3) See what non-Newtonian fluids do

Linear viscoelastic limit

\[
\lim_{\gamma_0 \to 0} G(t, \gamma_0) = G(t)
\]

At small strains the relaxation modulus is independent of strain.

The polystyrene solutions on the previous slide show time-strain independence, i.e. the curves have the same shape at different strains.

Damping function, \( h(\gamma_0) \):

\[
h(\gamma_0) \equiv \frac{G(t, \gamma_0)}{G(t)}
\]

The damping function summarizes the non-linear effects as a function of strain amplitude.

What types of materials generate stress in proportion to the strain imposed? Answer: elastic solids

Hooke's Law for elastic solids: \( \tau_{21}(t) = -G\gamma_{21}(0, t) \)

Similar to the linear spring law
1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a $g(t)$
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

Hookean solids

2) Predict what Newtonian fluids would do.

Hookean Solid

Constitutive Equation

$$\bar{\mathbf{e}} = -G\gamma(0, t)$$

Investigating Stress/Deformation Relationships (Rheology)

1) Choose a material function

1. Choice of flow (shear or elongation)

\[ y = \begin{pmatrix} \dot{\zeta}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \quad y = \begin{pmatrix} -\frac{1}{2} \dot{\varepsilon}(t)x_1 \\ -\frac{1}{2} \dot{\varepsilon}(t)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123} \]

2. Choice of time dependence of $\zeta(t)$ or $\dot{\varepsilon}(t)$

3. Material functions definitions: will be based on $\tau_{21}, N_1, N_2$ in shear or $\tau_{22} - \tau_{11}, \tau_{22} - \tau_{11}$ in elongational flows.
Summary of shear rate kinematics (part 2; strain-based)

1) Choose a material function – Strain based

**Small-Amplitude Oscillatory Shear Material Functions**

**Imposed Kinematics:**
\[
\gamma(t) = \gamma_0 \cos(\omega t)
\]
\[
\dot{\gamma}(t) = \gamma_0 \omega \sin(\omega t)
\]

\(N(t) = N'(t) = 0\) (linear viscoelastic regime)

**Material Stress Response:**
\[
\tau_{21}(t) = \tau_0 \sin(\omega t + \delta)
\]

\(\delta = \) phase difference between stress and strain waves

**Material Functions:**
\[
\tau_{21}/\gamma_0 = -\tau_{21}/\gamma_0 = \tau_0 \sin(\omega t + \delta) = G' \sin(\omega t) + G'' \cos(\omega t)
\]

Storage modulus \(G'(\omega) \equiv \frac{\tau_0}{\gamma_0} \cos(\delta)\)

Loss modulus \(G''(\omega) \equiv \frac{\tau_0}{\gamma_0} \sin(\delta)\)
What is the strain in small-amplitude oscillatory shear?

(let $t_{ref} = 0$

Answer: $\gamma_{21}(0, t) = \frac{\tau_0}{\omega} \sin \omega t$

In SAOS the strain amplitude is small, and a sinusoidal imposed strain induces a sinusoidal measured stress.

$$-\tau_{21}(t) = \tau_0 \sin(\omega t + \delta)$$

$$-\tau_{21}(t) = \tau_0 \sin(\omega t + \delta)$$

$$= \tau_0 \sin \omega t \cos \delta + \tau_0 \cos \omega t \sin \delta$$

$$= [\tau_0 \cos \delta] \sin \omega t + [\tau_0 \sin \delta] \cos \omega t$$

portion in-phase with strain

portion in-phase with strain-rate
1) Choose a material function – Strain based

\[ \delta \text{ is the phase difference between the stress wave and the strain wave} \]

\[ \gamma_{21}(0,t) = \gamma_0 \sin \omega t \]

\[ \tau_{21}(t) \]

SAOS Material Functions

\[ \frac{-\tau_{21}(t)}{\gamma_0} = \left[ \frac{\tau_0 \cos \delta}{\gamma_0} \right] \sin \omega t + \left[ \frac{\tau_0 \sin \delta}{\gamma_0} \right] \cos \omega t \]

For Newtonian fluids, stress is proportional to strain rate:

\[ \tau_{21} = -\mu \dot{\gamma}_{21} \]

\[ G' \quad G'' \]

\[ G'' \text{ is thus known as the viscous loss modulus. It characterizes the viscous contribution to the stress response.} \]
2) Predict what Newtonian fluids would do.

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( f(t) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

Answer: \( \delta = \frac{\pi}{2}; G'' = \mu \omega \)
1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( g(t) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

Hookean solids
2) Predict what Newtonian fluids would do.

Hookean Solid
Constitutive Equation

\[
\tau = -G\gamma(0, t)
\]

Answer: \( \delta = 0; G' = G \)
3) See what non-Newtonian fluids do

SAOS Moduli of a Polymer Melt

We have discussed six shear material functions;
Now, the equivalent \textbf{elongational} material functions

\[ \mathbf{v} = \left( \begin{array}{l} -\frac{1}{2} \xi(t) x_1 \\ -\frac{1}{2} \xi(t) x_2 \\ \xi(t) x_3 \end{array} \right) \]
Steady Elongational Flow Material Functions

Imposed Kinematics:
\[ \gamma(t) = \left( \begin{array}{c} \frac{1}{2} \dot{\varepsilon}(t)x_1 \\ -\frac{1}{2} \dot{\varepsilon}(t)x_2 \\ \dot{\varepsilon}(t)x_3 \end{array} \right) \]
\[ \dot{\varepsilon}(t) = \dot{\varepsilon}_0 = \text{constant} \]

Material Stress Response:
\[ \tau_{11}(t) - \tau_{22}(t) \]

Material Functions:
Elongational Viscosity
\[ \eta_e(\dot{\varepsilon}_0) = \frac{\dot{\varepsilon}_0 - \dot{\tau}_{11}}{\tau_0} = \frac{-(\tau_{33} - \tau_{11})}{\dot{\varepsilon}_0} \]
Alternatively, \[ \bar{n}(\dot{\varepsilon}_0) \]

1) Choose a material function – elongational flow

What is the strain in steady elongational flow?
(let \( t_{ref} = 0 \))

\[ \gamma_2 = ? \]

(to answer, review how strain was developed/defined for previous flows. . . )
1) Choose a material function – elongational flow

Path to strain for shear:

\[
\mathbf{r}(t_{\text{ref}}), \mathbf{r}(t) \rightarrow \mathbf{u} \rightarrow \nabla \mathbf{u} \rightarrow \gamma(t_{\text{ref}}, t)
\]

Try to follow for elongation.

\[
\mathbf{r}(t_{\text{ref}}) = \begin{pmatrix} x_1(t_{\text{ref}}) \\ x_2(t_{\text{ref}}) \\ x_3(t_{\text{ref}}) \end{pmatrix}, \quad \mathbf{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad \mathbf{u}(t_{\text{ref}}, t) \equiv \mathbf{r}(t) - \mathbf{r}(t_{\text{ref}}) = ?
\]

\[
\dot{\varepsilon}(t) = \begin{pmatrix} -\frac{1}{2} \dot{\varepsilon}(t)x_1 \\ -\frac{1}{2} \dot{\varepsilon}(t)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}, \quad \dot{\varepsilon}(t) = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix}, \quad \gamma \equiv \nabla \mathbf{u} + (\nabla \mathbf{u})^T = ?
\]

What is strain?

More generally:

Deformation (strain)

\[
\mathbf{r}(t_{\text{ref}}) = \begin{pmatrix} x_1(t_{\text{ref}}) \\ x_2(t_{\text{ref}}) \\ x_3(t_{\text{ref}}) \end{pmatrix}, \quad \mathbf{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}
\]

This vector keeps track of the location of a fluid particle as a function of time.
1) Choose a material function – elongational flow

What is the strain in steady elongational flow?

(let $t_{ref} = 0$

(coose $t_{ref}=0$)

$$\varepsilon(t_{ref}, t) = \int_{t_{ref}}^{t} \dot{\varepsilon}(t') dt'$$

$$= \dot{\varepsilon}_0 t$$

The strain imposed is proportional to time.

The ratio of current length to initial length is exponential in time.

Hencky strain

2) Predict what Newtonian fluids would do.

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a $\varepsilon(t)$
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

2) Predict what Newtonian fluids would do.

$\tau = -\mu \dot{\gamma}$

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3) See what non-Newtonian fluids do

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( g(y) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

Steady State
Elongation
Viscosity

Both tension thinning and thickening are observed.

Figure 6.60, p. 215
Munstedt; PS melt

Trouton ratio: 
\[ Tr = \frac{\bar{\eta}}{\eta_0} \]
Investigating Stress/Deformation Relationships (Rheology)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( g(\gamma) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

5) Predict the material function (with new \( \eta(\nu) \))

What does this model predict for steady elongational viscosity?

\[
\tau = -M(\dot{\gamma}_0) \left[ \nabla \nu + (\nabla \nu)^T \right]
\]

\[
M(\dot{\gamma}_0) = \begin{cases} 
M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\
\frac{m\dot{\gamma}_0^{n-1}}{n} & \dot{\gamma}_0 \geq \dot{\gamma}_c 
\end{cases}
\]

\[
\eta_e = ?
\]

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What if we make the following replacement?

\[ \dot{\gamma}_0 \rightarrow \frac{\partial v_1}{\partial x_2} \]

This at least can be written for any flow and it is equal to the shear rate in shear flow.

**Investigating Stress/Deformation Relationships (Rheology)**

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( \gamma(\dot{\gamma}) \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

**Observations**

- The model contains parameters that are specific to shear flow – makes it impossible to adapt for elongational or mixed flows
- Also, the model should only contain quantities that are independent of coordinate system (i.e. invariant)

We will try to salvage the model by replacing the flow-specific kinetic parameter with something that is frame-invariant and not flow-specific.
4) Hypothesize a constitutive equation

We will take out the shear rate and replace with the magnitude of the rate-of-deformation tensor (which is related to the second invariant of that tensor).

\[
\tau = -M \left( \dot{\gamma} \right) \left[ \nabla \dot{\gamma} + (\nabla \dot{\gamma})^T \right]
\]

\[
M \left( \dot{\gamma} \right) = \begin{cases} 
M_0 & \dot{\gamma} < \dot{\gamma}_c \\
M_0 \left( \frac{\dot{\gamma}}{\dot{\gamma}_c} \right)^{n-1} & \dot{\gamma} \geq \dot{\gamma}_c
\end{cases}
\]

(Hold that thought; finish the chapter)

1) Choose a material function – elongational flow

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \( \dot{\gamma} \)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.

The other elongational experiments are analogous to shear experiments (see text)

- Elongational stress growth
- Elongational stress cessation (nearly impossible)
- Elongational creep
- Step elongational strain
- Small-amplitude Oscillatory Elongation (SAOE) (Redundant with SAOS)
Imposed Kinematics:
\[
\dot{\mathbf{e}} \equiv \begin{pmatrix} \frac{1}{2} \dot{\varepsilon}(t)x_1 \\ -\frac{1}{2} \dot{\varepsilon}(t)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}
\]
\[\dot{\varepsilon}(t) = \begin{cases} 0 & t < 0 \\ \dot{\varepsilon}_0 & t \geq 0 \end{cases}\]

Material Stress Response:
\[\tau_{11}(t) - \tau_{22}(t)\]

Material Functions:
\(\eta^L(t, \dot{\varepsilon}_0) \equiv \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}_0} = \frac{\dot{\varepsilon}_0}{\dot{\varepsilon}_0}\)
Alternatively, \(\eta^N(t, \dot{\varepsilon}_0)\)

3) See what non-Newtonian fluids do

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids do
4. Hypothesize a \(f(t)\)
5. Predict the material function
6. Compare with what non-Newtonian fluids do
7. Reflect, learn, revise model, repeat.
3) See what non-Newtonian fluids do

Start-up of Steady Elongation

Fit to an advanced constitutive equation (12 mode pom-pom model)

What’s next?

Make better constitutive equations

1. Add invariants (replace \( \gamma_0 \), which is flow specific).
2. Make constitutive equations that reference flow in the past (not purely instantaneous)
3. Investigate strain

Be inspired by material behavior

1. Become informed on more rheological behavior
2. Get more of a feel for what is observed and when
Done with Material Functions.

Let's move on to Experimental Data.

Chapter 6: Experimental Data

Small-Amplitude Oscillatory Shear - Storage and Loss Moduli

Figure 6.30, p. 192 Plazek and O'Rourke: PS

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