Torsional Shear Flow: Parallel-plate and Cone-and-plate

Prof. Faith A. Morrison

Why do we need more than one method of measuring viscosity?

- At low rates, torques/pressures become low
- At high rates, torques/pressures become high; flow instabilities set in
We will look at two flows measurable in torsional shear: steady and small-amplitude oscillatory shear (SAOS)

a. Steady

\[ \dot{\gamma}(t) \]

\[ \gamma_{21}(0,t) \]

\[ \tau_{21}(t) \]

Material functions: \( \eta(\dot{\gamma}), \Psi_1(\dot{\gamma}), \Psi_2(\dot{\gamma}) \)

f. SAOS

\[ \dot{\gamma}(t) \sim \gamma_{21}(0,t) \]

\[ \tau_{21}(t) \sim \tau_{21}(t) \]

Material functions: \( G'(\omega), G''(\omega) \) or \( \eta'(\omega), \eta''(\omega) \)
**Cox-Merz Rule**

\[ \eta(\dot{\gamma}) = \eta^*(\omega) \left|_{\dot{\gamma}=\omega} \right. \]

\[ \eta^*(\omega) = \sqrt{\eta^3 + \eta^\#} \left/ \omega \right. \]

\[ \eta^\# = \frac{G^2 + \theta^2}{\omega} \]

\( \eta^*, \eta', \) or \( \eta \) (Pa s)

An empirical way to infer steady shear data from SAOS data.

**Figure 6.32, p. 193**
Venkataraman et al.; LDPE

---

**Steady Shear Flow Material Functions**

**Kinematics:**

\[
\mathbf{\gamma} = \begin{pmatrix}
\dot{\gamma}(t) x_2 \\
0 \\
0
\end{pmatrix}
\]

\( \dot{\gamma}(t) = \dot{\gamma}_0 = \text{constant} \)

**Material Functions:**

First normal-stress coefficient:

\[ \psi_1 = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} \]

Second normal-stress coefficient:

\[ \psi_2 = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} \]

\( \eta = \frac{-\tau_{21}}{\dot{\gamma}_0} \)

Viscosity
Torsional Parallel-Plate Flow - Viscosity

Measureables:
Torque T to turn plate
Rate of angular rotation W

Note: shear rate experienced by fluid elements depends on their r position.

By carrying out a Rabinowitsch-like calculation, we can obtain the stress at the rim (r=R).

\[ \tau_z \bigg|_{r=R} = -\frac{T}{2\pi R^3} \left[ 3 + \frac{d \ln(T) / 2\pi R^3}{d \ln \dot{\gamma}_R} \right] \]

Correction required

\[ \eta(\dot{\gamma}_R) = \frac{\tau_z \bigg|_{r=R}}{\dot{\gamma}_R} \]

Torsional Parallel-Plate Flow - correction

\[ \frac{T}{2\pi R^3} \text{ is a function of } \dot{\gamma}_R \]

\[ \text{slope} = \frac{\frac{d \ln(T)}{2\pi R^3}}{\frac{d \ln \dot{\gamma}_R}{\dot{\gamma}_R}} \]

\[ \dot{\gamma}_R = \frac{R\Omega}{H} \]

\[ \eta(\dot{\gamma}_R) = \frac{T / 2\pi R^3}{\dot{\gamma}_R} \left[ 3 + \frac{d \ln(T) / 2\pi R^3}{d \ln \dot{\gamma}_R} \right] \]
Torsional Cone-and-Plate Flow - Viscosity

Measureables:
Torque $T$ to turn cone
Rate of angular rotation $\Omega$

Note: the introduction of the cone means that shear rate is independent of $r$.

Since shear rate is constant everywhere, so is stress, and we can calculate stress from torque.

$$\tau_{0\phi} = \text{constant} = \frac{3T}{2\pi R^2}$$

$$\eta(\dot{\gamma}) = \frac{3T \Theta_0}{2\pi R^3 \Omega}$$

No corrections needed in cone-and-plate

---

Torsional Cone-and-Plate Flow – 1st Normal Stress

Measureables:
Normal thrust $F$

The total upward thrust of the cone can be related directly to the first normal stress coefficient.

$$F = 2\pi \int_{0}^{R} \Pi_{\theta} \left[ \Theta_0 \dot{\gamma} \right] r dr = -\pi R^2 p_{atm}$$

(see text pp404-5; also DPL pp522-523)

$$\Psi_1(\dot{\gamma}) = \frac{2F \Theta_0^2}{\pi R^2 \Omega^2}$$
Torsional Cone-and-Plate Flow – 2\textsuperscript{nd} Normal Stress

- Cone and Plate:
  \[ \Pi_{22} - p_0 = -(N_1 + 2M_1) \ln \left( \frac{r}{R} \right) - N_2 \]
  (see Bird et al., DPL)

- MEMS used to manufacture sensors at different radial positions

The Normal Stress Sensor System (NSS)

RheoSense Incorporated (www.rheosense.com)

RheoSense Incorporated
Comparison with other instruments

Monolithic rheometer plate fabricated using silicon micromachining technology and containing miniature pressure sensors for \(N_1\) and \(N_2\) measurements

Sung-Gil Baek
RheoSense, Incorporated, 2587 Venture Drive, Suite 106, St. Paul, Minnesota 55123

John J. Magda
Department of Chemical and Process Engineering, University of Utah, 40 South Central Campus Drive, Room 3700, Salt Lake City, Utah 84112

**Limits on Measurements:** Flow instabilities in rheology

conce and plate flow

Figures 6.7 and 6.8, p. 175 Hutton; PDMS

Small-Amplitude Oscillatory Shear Material Functions

**Kinematics:**

\[
\mathbf{\gamma} = \begin{pmatrix}
\dot{\gamma}(t) x_2 \\
0 \\
0
\end{pmatrix}
\]

\[
\dot{\gamma}(t) = \gamma_0 \cos \omega t
\]

\[
\gamma_0 = \frac{\ddot{\gamma}_0}{\omega}
\]

**Material Functions:**

\[
\frac{-\tau_{21}(t, \gamma_0)}{\gamma_0} = G' \sin \omega t + G'' \cos \omega t
\]

\[
G'(\omega) \equiv \frac{\tau_0}{\gamma_0} \cos \delta
\]

Storage modulus

\[
(\delta \text{ is the phase difference between stress and strain})
\]

\[
G''(\omega) \equiv \frac{\tau_0}{\gamma_0} \sin \delta
\]

Loss modulus

© Faith A. Morrison, Michigan Tech U.
What is the strain in this flow?

\[
\gamma_{21}(0,t) = \int_0^t \gamma_{21}(t')dt'
\]

\[
= \int_0^t \gamma_0 \cos \omega t' dt'
\]

\[
= \frac{\gamma_0}{\omega} \sin \omega t
\]

The strain amplitude is \(\gamma_0 = \frac{\gamma_0}{\omega}\).

Generating Small Amplitude Oscillatory Shear (SAOS)

Steady shear

\[b(t) = Vt = h\dot{\gamma}_0 t\]

\[V = \text{constant}\]

Small-amplitude oscillatory shear

\[b(t) = h\gamma_0 \sin \omega t = \frac{h\gamma_0}{\omega} \sin \omega t\]

\[V = \text{periodic}\]
In **SAOS** the strain amplitude is small, and a sinusoidal imposed strain induces a sinusoidal measured stress.

\[-\tau_{21}(t) = \tau_0 \sin(\omega t + \delta)\]

\[-\tau_{21}(t) = \tau_0 \sin(\omega t + \delta) = \tau_0 \sin \omega t \cos \delta + \tau_0 \cos \omega t \sin \delta = [\tau_0 \cos \delta] \sin \omega t + [\tau_0 \sin \delta] \cos \omega t\]

portion in-phase with strain
portion in-phase with strain-rate

\[\delta\] is the phase difference between the stress wave and the strain wave
SAOS Material Functions

\[
- \frac{\tau_{21}(t)}{\gamma_0} = \left[ \frac{\tau_0 \cos \delta}{\gamma_0} \right] \sin \omega t + \left[ \frac{\tau_0 \sin \delta}{\gamma_0} \right] \cos \omega t
\]

portion in-phase with strain

portion in-phase with strain-rate

\[G' \quad G''\]

For Newtonian fluids, stress is proportional to strain rate:

\[\tau_{21} = -\mu \dot{\gamma}_{21}\]

\[G''\]

is thus known as the viscous loss modulus. It characterizes the viscous contribution to the stress response.

What types of materials generate stress in proportion to the strain imposed? Answer: elastic solids

Hooke’s Law for elastic solids

\[\tau_{21} = -G \dot{\gamma}_{21}\]

Similar to the linear spring law
SAOS Material Functions

\[-\frac{\tau_{21}(t)}{\gamma_0} = \left[ \frac{\tau_0 \cos \delta}{\gamma_0} \right] \sin \omega t + \left[ \frac{\tau_0 \sin \delta}{\gamma_0} \right] \cos \omega t\]

portion in-phase with strain

\[G'\]

portion in-phase with strain-rate

\[G''\]

For Hookean solids, stress is proportional to strain:

\[\tau_{21} = -G \gamma_{21}\]

\[G'\] is thus known as the elastic storage modulus. It characterizes the elastic contribution to the stress response.

(note: SAOS material functions may also be expressed in complex notation. See pp. 156-159 of Morrison, 2001)

SAOS Material Functions

\[|G^*| = \sqrt{G'^2 + G''^2}\]

\[\tan \delta = \frac{G'}{G''}\]

\[\eta' = \frac{G''}{\omega}\]

\[\eta'' = \frac{\eta^*}{\omega}\]

\[|\eta^*| = \sqrt{\eta'^2 + \eta''^2}\]

\[\|J^*\| = \frac{1}{G^*}\]

\[J' = \frac{1}{G'}\left(1 + \tan^2 \delta\right)\]

\[J'' = \frac{1}{G''}\left(1 + \left(\tan^2 \delta\right)^{-1}\right)\]

© Faith A. Morrison, Michigan Tech U.
Assignment:

For the PDMS polymer in the lab

• measure and report on the true steady shear viscosity at room temperature, as a function of shear rate, as measured with the torsional cone-plate rheometer

• Report G' and G'' as a function of frequency at room temperature.

• Check to see if the Cox-Merz rule holds for PDMS.

\[
\eta'(\dot{\gamma}) = \left| \eta^*(\omega) \right|_{\dot{\gamma}=\omega}
\]

• If you would like to do a bit more experimentation, also report these quantities at 35 and 50°C and perform time-temperature superposition; this is optional.