CM3310 Spring 2008

(Dr. Tom Co, 2/10/2008)

Lecture 8. Numerical Methods for Solving ODEs

1. Euler Method

Given:

a. Differential equations,

$$\frac{dx}{dt} = f(t,x)$$

b. Initial conditions, $x(0) = x_o$

User Specify:

a. Time interval, $\Delta t = t_{k+1} - t_k$ (usually assumed fixed and uniform)
b. Final time, $t_{final} = N\Delta t$

Procedure:

Step 1: Replace derivatives by difference approximation,

$$\frac{dx}{dt} \cong \frac{\Delta \mathrm{x}}{\Delta t} = \frac{x_{k+1} - x_k}{\Delta t}$$

Step 2: Rearrange equations such that future values of the states can be evaluated using current values

$$\frac{x_{k+1} - x_k}{\Delta t} = f(t_k, x_k)$$

$$x_{k+1} = x_k + \Delta t\, f(t_k, x_k)$$

Step 3: Continue iteration until specified final time

$$x_1 = x_o + \Delta t\, f(0, x_o)$$

$$x_2 = x_1 + \Delta t\, f(t_1, x_1)$$

$$\vdots$$

$$x_N = x_{N-1} + \Delta t\, f(t_{N-1}, x_{N-1})$$

Example:

$$\frac{dh}{dt} = 2 - \sqrt{h} \quad ; \quad h(0) = 1.0$$

Suppose we chose $\Delta t = 0.1$, and $t_{final} = 200\Delta t = 20$. Then the iterative equations are given by,

$$h_{k+1} = h_k + 0.1\left(2 - \sqrt{h_k}\right)$$

Thus, we can iterate starting with initial condition, $h_o = 1.0$.

$$h_1 = h_o + 0.1\left(2 - \sqrt{h_o}\right) = 1.1000$$
$$h_2 = h_1 + 0.1\left(2 - \sqrt{h_1}\right) = 1.1951$$
$$\vdots$$
$$h_{19} = h_{18} + 0.1\left(2 - \sqrt{h_{18}}\right) = 3.9844$$
$$h_{20} = h_{19} + 0.1\left(2 - \sqrt{h_{19}}\right) = 3.9847$$

2. Runge-Kutta Method

Procedure:

Step 1: Evaluate intermediate corrections $\delta_1, \delta_2, \delta_3, \delta_4$:

$$\delta_1 = \Delta t\, f(t_k, x_k)$$
$$\delta_2 = \Delta t\, f\left(t_k + \frac{\Delta t}{2}, x_k + \frac{\delta_1}{2}\right)$$
$$\delta_3 = \Delta t\, f\left(t_k + \frac{\Delta t}{2}, x_k + \frac{\delta_2}{2}\right)$$
$$\delta_4 = \Delta t\, f(t_k + \Delta t, x_k + \delta_3)$$

Step 2: Add a weighted average of the intermediate corrections to current value:

$$x_{k+1} = x_k + \left(\frac{\delta_1}{3} + \frac{\delta_2}{6} + \frac{\delta_3}{6} + \frac{\delta_4}{3}\right)$$

Example:

$$\frac{dh}{dt} = 2 - \sqrt{h} \quad ; \quad h(0) = 1.0$$

k=1 :

$$\delta_1 = \Delta t\left(2 - \sqrt{h_o}\right) = 0.1$$
$$\delta_2 = \Delta t\left(2 - \sqrt{h_o + \frac{\delta_1}{2}}\right) = 0.09753$$
$$\delta_3 = \Delta t\left(2 - \sqrt{h_o + \frac{\delta_2}{2}}\right) = 0.09759$$
$$\delta_4 = \Delta t\left(2 - \sqrt{h_o + \delta_3}\right) = 0.09523$$

$$h_1 = h_o + \left(\frac{\delta_4}{3} + \frac{\delta_2}{6} + \frac{\delta_3}{6} + \frac{\delta_4}{3}\right) = 1.097598$$
$$\vdots$$

3. Issues/Remarks:

- For the same specified Δt, the Runge-Kutta method gives four-order of magnitude improvement in accuracy.
- A variation on the Euler method above is called "Euler-backward" method.
$$x_{k+1} = x_k + \Delta t\, f(t_{k+1}, x_{k+1})$$
Which is called an "implicit" method. This variation usually needs solution of nonlinear equations. However, implicit methods are usually more stable.
- A popular variation of Runge Kutta method uses the "error control" approach to adapt the step size, Δt_k. One version is to compare the results using both a fourth and fifth order method. If the difference is significant, then the step size is reduced.
- When handling multivariable or high order cases, one needs to put the equations into a set of first order differential equations.
- For implementation in Excel, check out the link: http://www.chem.mtu.edu/~tbco/cm416/RKTutorial.html