CM3310 Spring 2008

(Dr. Tom Co, 2/18/2008)

Lecture 11. Transfer Functions 1

1. Definition (page 95)

A process transfer function, $G(s)$, is the Laplace domain relationship between an input $\hat{u}(s)$ and output $\hat{y}(s)$ of a process (assuming zero initial conditions).

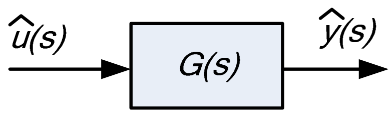


Figure 1. Block diagram.

2. Transfer Functions from Differential Equations

First Order Process:

$$\tau \frac{dy}{dt} + y = K\,u$$

$$\tau(s\,\hat{y}) + \hat{y} = K\hat{u}$$

$$\hat{y} = \left(\frac{K}{\tau s + 1}\right)\hat{u}$$

Thus,

$$G(s) = \frac{K}{\tau s + 1}$$

First Order Process with Time Delay (FOPTD) :

$$\tau \frac{dy}{dt} + y = K\,u\left(t - \tau_{\text{delay}}\right)$$

$$\tau(s\,\hat{y}) + \hat{y} = Ke^{-\tau_{\text{delay}} s}\,\hat{u}$$

$$\hat{y} = \left(\frac{Ke^{-\tau_{\text{delay}} s}}{\tau s + 1}\right)\hat{u}$$

Thus,

$$G(s) = \frac{Ke^{-\tau_{\text{delay}} s}}{\tau s + 1}$$

Second Order Process:

$$\tau_n{}^2 \frac{d^2y}{dt^2} + 2\tau_n\zeta \frac{dy}{dt} + y = K\,u$$

$$\hat{y} = \left(\frac{K}{\tau_n{}^2 s^2 + 2\tau_n\zeta s + 1} \right) \hat{u}$$

Thus,

$$G(s) = \frac{K}{\tau_n{}^2 s^2 + 2\tau_n\zeta s + 1}$$

General n^{th} Order Process:

$$\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \cdots + b_1\frac{du}{dt} + b_0 u$$

$$\hat{y} = \left(\frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \right) \hat{u}$$

Equivalent set of ODEs: (useful for simulations)

$$\frac{dx_1}{dt} = -a_{n-1}x_1 + x_2 + b_{n-1}u$$
$$\frac{dx_2}{dt} = -a_{n-2}x_1 + x_3 + b_{n-2}u$$
$$\vdots$$
$$\frac{dx_{n-1}}{dt} = -a_1 x_1 + x_n + b_1 u$$
$$\frac{dx_n}{dt} = -a_0 x_1 + b_0 u$$
$$y = x_1$$

Example:

$$\frac{dx_1}{dt} = -3x_1 + x_2 + u$$
$$\frac{dx_2}{dt} = -2x_1 + x_3 - 2u$$
$$\frac{dx_3}{dt} = -2x_1 + 3u$$
$$y = x_1$$

$$\rightarrow \quad \hat{y} = \left(\frac{s^2 - 2s + 3}{s^3 + 3s^2 + 2s + 2} \right) \hat{u}$$

3. Transfer Functions of P, PI and PID controllers: $G_c(s)$

P	K_c
PI	$K_c\left(\frac{\tau_{int}s+1}{\tau_{int}s}\right)$
PID (ideal)	$K_c\left(\frac{\tau_{int}s+1}{\tau_{int}s}+\tau_{der}s\right)$
PID (real)	$K_c\left(\frac{\tau_{int}s+1}{\tau_{int}s}\right)\left(\frac{\tau_{der}s+1}{\alpha\tau_{der}s+1}\right)$; $\alpha < 0.1$

4. Analysis of Transfer Functions: (page 111)

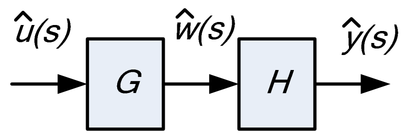
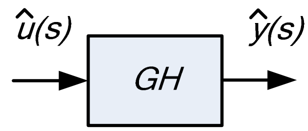
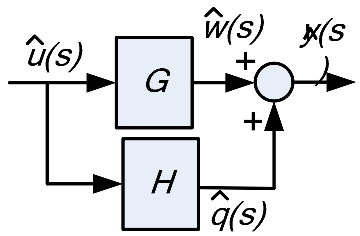
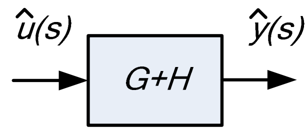
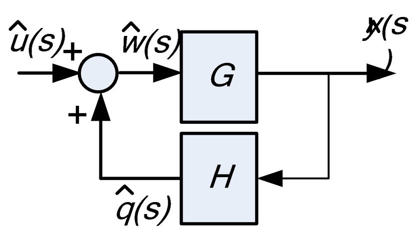
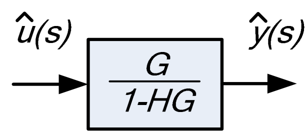
$$G(s)=\frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=\frac{k_p(s-z_1)\cdots(s-z_{n-1})}{(s-p_1)\cdots(s-p_n)}$$

a) The roots of the denominator: $p_1, p_2, \cdots, p_n$ are known as the "poles" of $G(s)$.
b) The roots of the numerator: $z_1, z_2, \cdots, z_{n-1}$ are known as the "zeroes" of $G(s)$.
c) The term k_p is the "process gain".
d) The poles are equivalent to the eigenvalues of the process, thus the transfer function $G(s)$ is unstable if any of the poles has a positive real part.
e) If any of the zeros has a positive real part, the process will exhibit non-minimum phase response.

5. Typical Input Response Tests

Step Test	$\hat{u}=\frac{\alpha}{s}$
Impulse Test	$\hat{u}=\alpha$
Sinusoidal Test	$\hat{u}=A\,\frac{\alpha}{s^2+\alpha^2}$

6. Equivalent Transfer Functions using Block Algebra

Series		
Parallel		
Feedback		

General guidelines to find equivalent transfer functions:

i) Name all signal lines
ii) Tapped lines can be named the same
iii) A negative entry to summing junctions can be replaced by a positive entry with a block of -1.
iv) Start with the output variable and write all the algebraic equations until the input variable is obtained
v) If more than one external input exists, one can use superposition.

Drills:

1. For the process described by

$$\frac{dC}{dt} = -2C + 3T + 2u$$
$$\frac{dT}{dt} = 1.5C - 2.2T - 3.6u$$

   a) Find the transfer function from u to C.
   b) Find the transfer function form u to T.
   c) Draw a block diagram of the process.

2. Obtain a set of first order differential equations that would yield the following transfer function equation:

$$\hat{y} = \left(\frac{2s^2 + 4}{s^3 + 3s^2 + 2s + 2.5}\right)\hat{u}$$

3. Find the zeros and poles of $G(s)$ and determine the stability:

$$G(s) = \frac{s + 1}{s^2 + 4s + 4}$$

4. Obtain the equivalent transfer function from $\hat{u}$ to $\hat{y}$ for Figure 2.

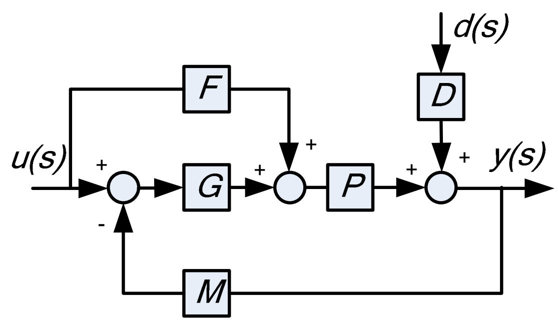
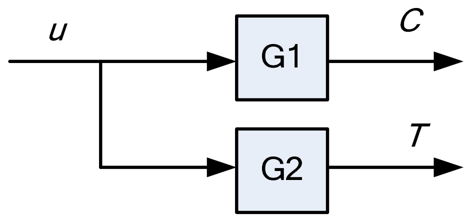


Figure 2. Feedback Control

Answers:

1. $\hat{C} = \left(\frac{2s-6.4}{s^2+4.2s-0.1}\right)\hat{u}$; $\hat{T} = \left(\frac{-4.2s-3.6}{s^2+4.2s-0.1}\right)\hat{u}$



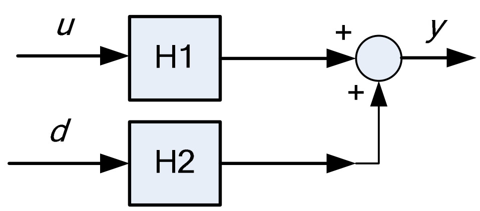
where $G_1 = \left(\frac{2s-6.4}{s^2+4.2s-0.1}\right)$ $and\ G_2 = \left(\frac{-4.2s-3.6}{s^2+4.2s-0.1}\right)$.

2.

$$\frac{dx_1}{dt} = -3x_1 + x_2 + 2u$$
$$\frac{dx_2}{dt} = -2x_1 + x_3$$
$$\frac{dx_3}{dt} = -2.5x_1 + 4u$$
$$y = x_1$$

3. zeros: -1 , poles: -2, -2

4.



Where,

$$H1 = \frac{GP + FP}{1 + MGP}$$
$$H2 = \frac{D}{1 + MGP}$$