CM3310 Spring 2008

(Dr. Tom Co, 3/18/2008)

Lecture 15. Introduction to Frequency Response Methods

1. **Motivation**
   a) One approach to modeling complex process
   b) A different perspective to signal analysis and filtering
   c) Introduces a different stability criterion
   d) Allows robustness in controller designs

2. **Frequency Response Experiment (** see page 216-219 **)**

   Given: A process with one input, $u(t)$, and one output, $y(t)$.

   Procedure:

   Choose u_{bias}, a range of frequencies: $\omega_1, \omega_2, \dots, \omega_n$
   and amplitude: $A_1, A_2, \dots, A_n$.

   (Often, one uses $A_1 = A_2 = \cdots = A_n$, and $u_{bias} = 0$).

   For each frequency:

   Step 1. Introduce the sinusoidal input (see Figure 1)

   $$u_k(t) = u_{bias} + A_k \sin(\omega_k t)$$

   Step 2. Wait for the output to attain periodic oscillation (see Figure 2)

   $$y_k(t) = y_{k,Transient}(t) + y_{k,Periodic}(t)$$
   $$y_{k,Periodic}(t) = B_k \sin(\omega_k t + \phi_k)$$

   where , ϕ_k is known as the "phase shift" calculated as

   $$\phi_k = -2\pi \frac{\Delta P}{P} \text{(in radians)} = -360° \frac{\Delta P}{P} \text{(in degrees)}$$
   $$= -\omega\, \Delta P \text{(in radians)} = -\omega\, \Delta P \frac{180°}{2\pi} \text{(in degrees)}$$

   Step 3. Record the following: $A_k, \omega_k, B_k, \phi_k$

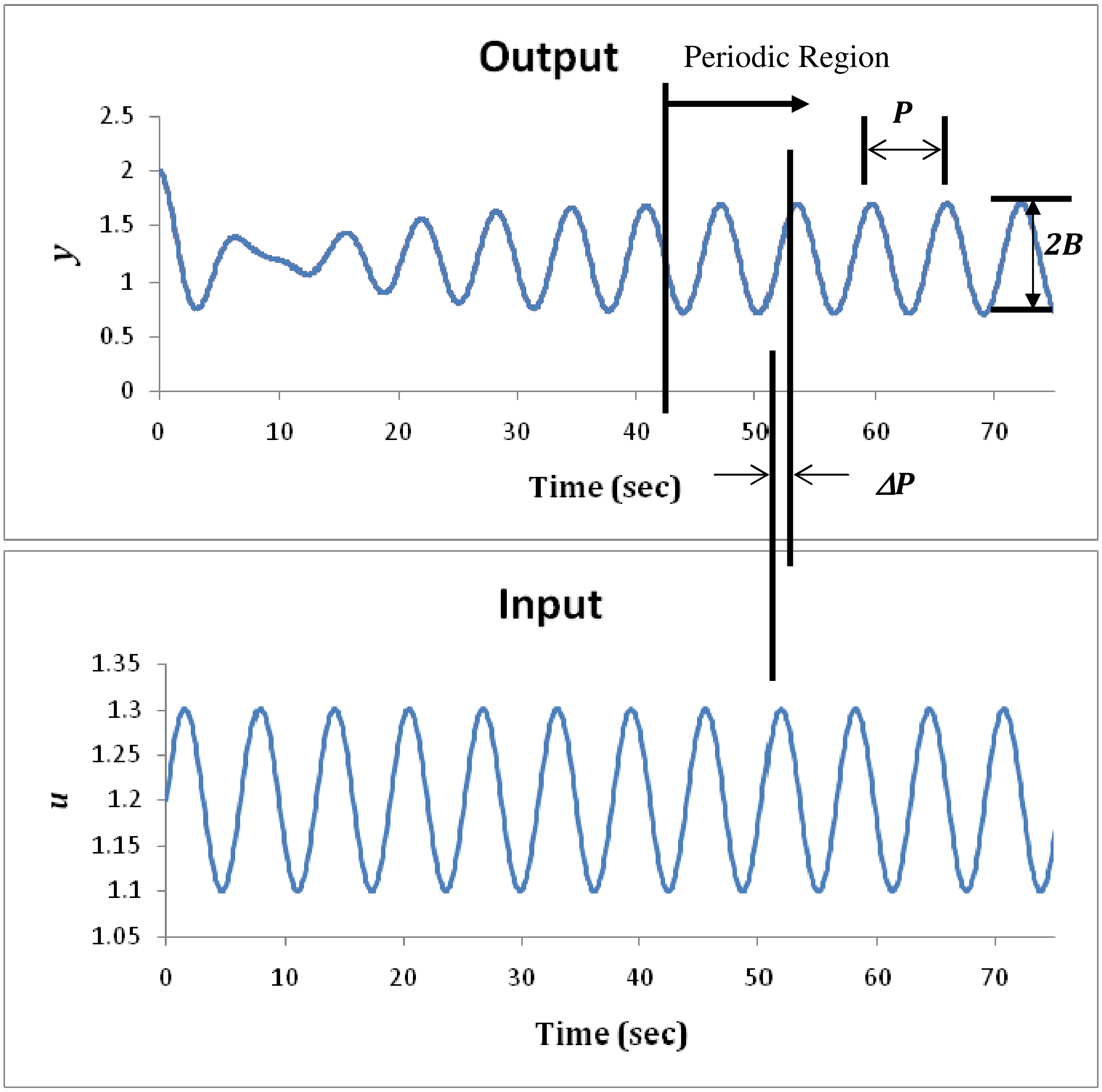


Figure 1. Sample frequency response experiment.

Example: (from plot in Figure 1) $A = 0.1$, $\omega = 1$ rad/sec

$$u = 1.2 + 0.1\sin(1t)$$

From figure, $\Delta P = 1.6$ sec. , $B = \frac{0.989}{2} = 0.4945$, thus

$$\phi = -1\left(\frac{\text{rads}}{\text{sec}}\right) \cdot 1.6(\text{secs}) = -1.6 \text{ rads} = -91.7°$$

$$y_{periodic} = 0.4945 \sin(1t - 1.6)$$

3. Frequency Data Representations

a) Data table

Input Pars (chosen)		Output Pars (measured)		Bode Plot Data			Nyquist Plot Data	
A	ω (sec^{-1})	B	ϕ (rads)	AR	LM (dB)	ϕ (deg)	ReG	ImG
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.1	1	0.4945	-1.6	4.945	13.88	-91.67	-1.123	-4.943
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

where, AR is known as the amplitude ratio, LM is the log modulus, ReG is the real component and ImG is the imaginary component, calculated as follows:

$$AR = \frac{A}{B}$$
$$LM = 20\log(AR)$$
$$ReG = AR\cos(\phi)$$
$$ImG = AR\sin(\phi)$$

b) Bode plots:

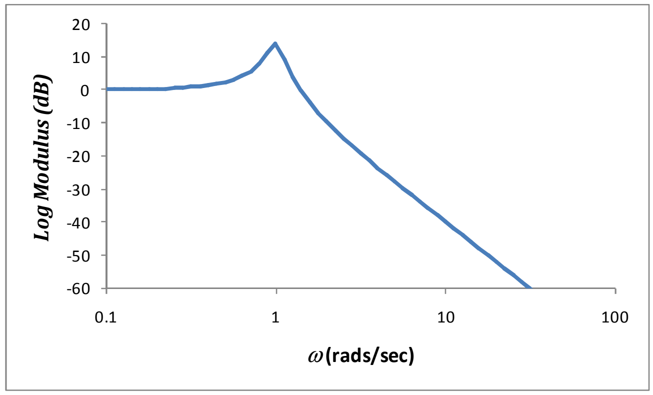
- 

Figure 2a. Bode plot type I: Log modulus vs. frequency

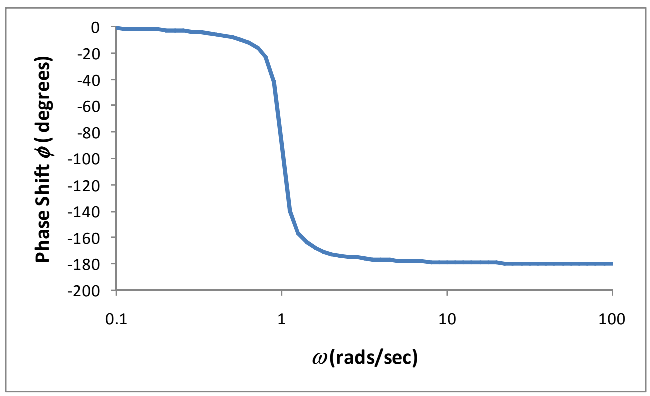
- 

Figure 2b. Bode plot type II: Phase shift vs. frequency

c) Nyquist plot:

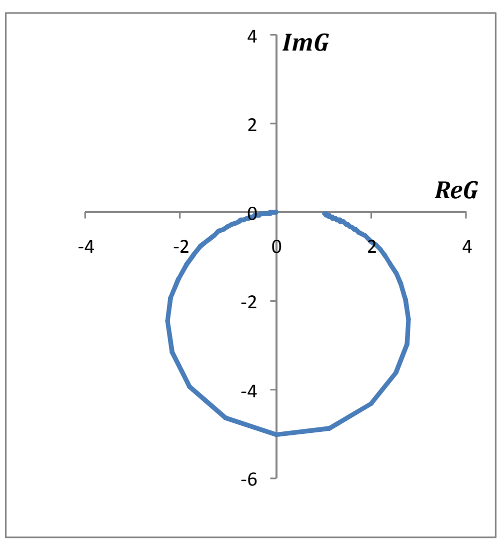


Figure 3. Nyquist plot: ImG vs. ReG

Connections: (see page 220-221 for proof)

Suppose the transfer function of the process is given by $G(s)$,

a) Transfer Functions with Nyquist Plot Data

$$ReG = \text{Real}(\, G\,(i\omega)\,)$$
$$ImG = \text{Imag}(\, G(i\omega))$$

b) Bode Plot Data with Transfer functions and Nyquist Plot Data

$$AR = |G(i\omega)| = \sqrt{ReG^2 + ImG^2}$$
$$\phi = \arg\big(G(i\omega)\big) = \tan^{-1}\left(\frac{ReG}{ImG}\right)$$

Example:

$$\begin{aligned}
G(s) &= \frac{1}{\tau_n^2 s^2 + 2\zeta\tau_n s + 1} \\
G(i\omega) &= \frac{1}{\tau_n^2 (i\omega)^2 + 2\zeta\tau_n (i\omega) + 1} \\
&= \frac{1}{(1-\tau_n^2\omega^2) + i(2\zeta\tau_n\omega)} \cdot \frac{(1-\tau_n^2\omega^2) - i(2\zeta\tau_n\omega)}{(1-\tau_n^2\omega^2) - i(2\zeta\tau_n\omega)} \\
&= \left[\frac{(1-\tau_n^2\omega^2)}{(1-\tau_n^2\omega^2)^2 + (2\zeta\tau_n\omega)^2}\right] - i\left[\frac{(2\zeta\tau_n\omega)}{(1-\tau_n^2\omega^2)^2 + (2\zeta\tau_n\omega)^2}\right]
\end{aligned}$$

$$ReG = \left[\frac{(1-\tau_n^2\omega^2)}{(1-\tau_n^2\omega^2)^2 + (2\zeta\tau_n\omega)^2}\right]$$
$$ImG = -\left[\frac{(2\zeta\tau_n\omega)}{(1-\tau_n^2\omega^2)^2 + (2\zeta\tau_n\omega)^2}\right]$$

$$AR = |G(i\omega)| = \frac{1}{\sqrt{(1-\tau_n^2\omega^2)^2 + (2\zeta\tau_n\omega)^2}}$$
$$LM = 20\log((1-\tau_n^2\omega^2)^2 + (2\zeta\tau_n\omega)^2)$$

$$\phi = \tan^{-1}\left(\frac{ImG}{ReG}\right) = -\left[\tan^{-1}\left(\frac{2\zeta\tau_n}{1-\tau_n^2\omega^2}\right)\right]\cdot\frac{180°}{\pi} \quad \text{(in degrees)}$$

Let $\tau_n = 1$ and $\zeta = 0.1$, then

$$AR(\omega) = \frac{1}{\sqrt{(1-\omega^2)^2 + (0.2\omega)^2}} = \frac{1}{\sqrt{1 - 1.96\omega^2 + \omega^4}}$$

$$LM(\omega) = -10\log(1 - 1.96\omega^2 + \omega^4)$$

$$\phi(\omega) = -\left[\tan^{-1}\left(\frac{0.2}{1-\omega^2}\right)\right]\cdot\frac{180°}{\pi}$$

$$ReG = \left[\frac{1-\omega^2}{1 - 1.96\omega^2 + \omega^4}\right]$$

$$ImG = -\left[\frac{0.2\omega}{1 - 1.96\omega^2 + \omega^4}\right]$$

then the Bode plots and Nyquist plots for this case is shown in Figure 2a, 2b and 3.

Drill: Find the log modulus and phase shift, as functions of frequency, of the transfer function

$$G(s) = \frac{-2s+1}{(s+1)(2s+1)}$$

Answer to Drill:

$$LM = -10\log(1+\omega^2) \quad ; \quad \phi = \tan^{-1}\left(\frac{4\omega^3 - 5\omega}{1 - 8\omega^2}\right)$$