Solution to Exam 1, Feb.6 2001

1. $$\frac{dh}{dt} = \frac{F_o - k_v \cdot u \cdot \sqrt{h}}{\pi \cdot r^2} \qquad r^2 = 2 \cdot R \cdot h - h^2$$

$$\frac{dh}{dt} = \frac{F_o - k_v \cdot u \cdot \sqrt{h}}{\pi \cdot \left(2 \cdot R \cdot h - h^2\right)}$$

given: $F_{oss} := 2 \cdot \frac{ft^3}{min}$ $\quad u_{ss} := 0.5$ $\quad R := 20 \cdot ft$ $\quad k_v := 1.3 \cdot \frac{ft^3}{min} \cdot \frac{1}{\sqrt{ft}}$

a) Steady State: $h_{ss} := \left(\frac{F_{oss}}{k_v \cdot u_{ss}}\right)^2 \qquad h_{ss} = 9.467 \, ft$

b) Linearization

$$f\left(h, u, F_o\right) = \frac{F_o - k_v \cdot u \cdot \sqrt{h}}{\pi \cdot \left(2 \cdot R \cdot h - h^2\right)}$$

$$\frac{df}{dh} = \frac{-1}{2} \cdot k_v \cdot \frac{u}{\left[\sqrt{h} \cdot \left[\pi \cdot \left(2 \cdot R \cdot h - h^2\right)\right]\right]} - \frac{\left(F_o - k_v \cdot u \cdot \sqrt{h}\right)}{\left[\pi \cdot \left(2 \cdot R \cdot h - h^2\right)^2\right]} \cdot (2 \cdot R - 2 \cdot h)$$

$$\frac{df}{du} = -k_v \cdot \frac{\sqrt{h}}{\left[\pi \cdot \left(2 \cdot R \cdot h - h^2\right)\right]}$$

$$\frac{df}{dF_o} = \frac{1}{\left[\pi \cdot \left(2 \cdot R \cdot h - h^2\right)\right]}$$

linearized model: $\frac{dh}{dt} = \alpha + \beta \cdot \left(h - h_{ss}\right) + \gamma \cdot \left(u - u_{ss}\right) + \delta \cdot \left(F_o - F_{oss}\right)$

$$\alpha := \frac{F_{oss} - k_v \cdot u_{ss} \cdot \sqrt{h_{ss}}}{\pi \cdot \left(2 \cdot R \cdot h_{ss} - {h_{ss}}^2\right)} \qquad \alpha = 0 \, \frac{ft}{min}$$

$$\beta := \frac{-1}{2} \cdot k_v \cdot \frac{u_{ss}}{\left[\sqrt{h_{ss}} \cdot \left[\pi \cdot \left(2 \cdot R \cdot h_{ss} - {h_{ss}}^2\right)\right]\right]} - \frac{\left(F_{oss} - k_v \cdot u_{ss} \cdot \sqrt{h_{ss}}\right)}{\left[\pi \cdot \left(2 \cdot R \cdot h_{ss} - {h_{ss}}^2\right)^2\right]} \cdot \left(2 \cdot R - 2 \cdot h_{ss}\right)$$

$$\beta = -1.163 \cdot 10^{-4} \, min^{-1}$$

$$\gamma := -k_v \cdot \frac{\sqrt{h_{ss}}}{\left[\pi \cdot \left(2 \cdot R \cdot h_{ss} - {h_{ss}}^2\right)\right]} \qquad \gamma = -4.405 \cdot 10^{-3} \, \frac{ft}{min}$$

$$\delta := \frac{1}{\left[\pi\cdot\left(2\cdot R\cdot h_{ss} - h_{ss}^{2}\right)\right]} \qquad \delta = 1.101\cdot 10^{-3}\circ ft^{-2}$$

2. Given: $$10\cdot\frac{dT_1}{dt} = \left(100 - T_1\right) + 10\cdot T_2$$

$$5\cdot\frac{dT_2}{dt} = \left(T_1 - T_2\right) + 2\cdot u$$

$$u = 0.5 + k_c\cdot\left(30 - T_2\right)$$

$$10\cdot\frac{dT_1}{dt} = \left(100 - T_1\right) + 10\cdot T_2 \qquad \text{(eq1)}$$

$$5\cdot\frac{dT_2}{dt} = \left(T_1 - T_2\right) + 2\cdot\left[0.5 + k_c\cdot\left(30 - T_2\right)\right] \qquad \text{(eq2)}$$

differentiate eq1

$$10\cdot\frac{d^2}{dt^2}T_1 = \frac{-dT_1}{dt} + 10\cdot\frac{dT_2}{dt}$$

substitute using eq2

$$10\cdot\frac{d^2}{dt^2}T_1 = \frac{-dT_1}{dt} + \left(-4.\cdot k_c - 2.\right)\cdot T_2 + 120.\cdot k_c + 2.\cdot T_1 + 2. \qquad \text{(eq3)}$$

obtain T2 from eq1

$$T_2 = \frac{dT_1}{dt} - \frac{100 - T_1}{10}$$

and substitue this into eq3,

$$10\cdot\frac{d^2}{dt^2}T_1 + \left(3 + 4\cdot k_c\right)\cdot\frac{d}{dt}T_1 + \left(0.4\cdot k_c - 1.8\right)\cdot T_1 = 22 + 160\cdot k_c$$

a) Stability: (need all the coefficients to be of the same sign) $k_c > \frac{1.8}{0.4} = 4.5$

b) Offset:T2set-T2ss $0 = \left(100 - T_{1ss}\right) + 10\cdot T_{2ss}$

$$T_{1ss} = 100 + 10\cdot T_{2ss}$$

$$0 = \left(T_{1ss} - T_{2ss}\right) + 2\cdot\left[0.5 + k_c\cdot\left(30 - T_{2ss}\right)\right]$$

$$T_{2ss} = \frac{\left(101. + 60.\cdot k_c\right)}{\left(-9. + 2.\cdot k_c\right)}$$

$$\text{Offset}\left(k_c\right) := 30 - \frac{\left(101. + 60.\cdot k_c\right)}{\left(-9. + 2.\cdot k_c\right)} \qquad \text{Offset}(30) = -7.275$$

3. Given: $\frac{dP}{dt} = -3 \cdot P + 2 \cdot u$

$$u = 10 + k_c \cdot \left[(P_{set} - P) + \frac{1}{\tau_I} \cdot \left[\int (P_{set} - P) \, dt \right] \right]$$

$$\frac{d^2}{dt^2} P + (2 \cdot k_c + 3) \cdot \frac{dP}{dt} + \left(\frac{2 \cdot k_c}{\tau_I} \right) \cdot P = \left(\frac{2 \cdot k_c}{\tau_I} \right) \cdot P_{set}$$

$$\frac{\tau_I}{2 \cdot k_c} \cdot \left(\frac{d^2}{dt^2} P \right) + \left(\frac{2 \cdot k_c + 3}{2 \cdot k_c} \cdot \tau_I \right) \cdot \frac{dP}{dt} + P = P_{set}$$

Comparing with the standard form of 2nd order process,

$$\tau_n^{2} \cdot \left(\frac{d^2}{dt^2} P \right) + 2 \cdot \tau_n \cdot \zeta \cdot \frac{d}{dt} P + P = P_{set}$$

$$\tau_n = \sqrt{\frac{\tau_I}{2 \cdot k_c}}$$

$$\zeta = \frac{1}{2} \cdot \sqrt{\frac{2 \cdot k_c}{\tau_I}} \cdot \left(\frac{2 \cdot k_c + 3}{2 \cdot k_c} \cdot \tau_I \right)$$

with kc=1, for overdamped process we need :

$$\frac{5}{4} \cdot \sqrt{2} \cdot \sqrt{\tau_I} > 1 \qquad \text{or} \qquad \tau_I > \frac{8}{25} = 0.32$$

4. (bonus) Given : $(1 + z) \cdot \frac{d^3}{dt^3} z + 3 \cdot \left(\frac{d}{dt} z \right)^3 = \frac{d^2}{dt^2} z$

let $y = \frac{dz}{dt}$ $\qquad x = \frac{d^2}{dt^2} z$

With fixed Δt=0.1, the recursion equations under Euler's method is given by,

$$z_{k+1} = z_k + 0.1 \cdot y_k$$

$$y_{k+1} = y_k + 0.1 \cdot x_k$$

$$x_{k+1} = x_k + 0.1 \cdot \left(\frac{1}{1 + z_k} \right) \cdot \left[x_k - 3 \cdot (y_k)^3 \right]$$