Solution to Exam 2 (Feb 8, 1993)

1. Taking Laplace transform of both sides, with zero intial conditions,

$$P(s) = \left(\frac{0.2}{s+0.12}\right)\cdot S(s) + \left(\frac{0.1}{s+0.12}\right)\cdot L(s) - \left(\frac{0.5}{s+0.12}\right)\cdot V(s)$$

Using PI control of P by manipulating V,

$$V(s) = Kc\cdot\left(1 + \frac{1}{10\cdot s}\right)\cdot e(s) = \left[\frac{Kc}{10}\cdot\frac{(10\cdot s+1)}{s}\right]\cdot e(s)$$

$$e(s) = P_{set}(s) - P(s)$$

After substitution,

$$P(s) = \left(\frac{0.2}{s+0.12}\right)\cdot S(s) + \left(\frac{0.1}{s+0.12}\right)\cdot L(s) - \left(\frac{0.5}{s+0.12}\right)\cdot\left[\frac{Kc}{10}\cdot\frac{(10\cdot s+1)}{s}\right]\cdot\left(P_{set}(s) - P(s)\right)$$

$$\left[1 - \left(\frac{0.5}{s+0.12}\right)\cdot\left[\frac{Kc}{10}\cdot\frac{(10\cdot s+1)}{s}\right]\right]\cdot P(s)$$
$$= \left(\frac{0.2}{s+0.12}\right)\cdot S(s) + \left(\frac{0.1}{s+0.12}\right)\cdot L(s) - \left(\frac{0.5}{s+0.12}\right)\cdot\left[\frac{Kc}{10}\cdot\frac{(10\cdot s+1)}{s}\right]\cdot P_{set}(s)$$

$$\left[\frac{s^2 + (.12 - .5\cdot Kc)\cdot s - 5.0\cdot 10^{-2}\cdot Kc}{s\cdot(s+0.12)}\right]\cdot P(s)$$
$$= \left(\frac{0.2}{s+0.12}\right)\cdot S(s) + \left(\frac{0.1}{s+0.12}\right)\cdot L(s) - \left(\frac{0.5}{s+0.12}\right)\cdot\left[\frac{Kc}{10}\cdot\frac{(10\cdot s+1)}{s}\right]\cdot P_{set}(s)$$

$$\left[s^2 + (.12 - .5\cdot Kc)\cdot s - 5.0\cdot 10^{-2}\cdot Kc\right]\cdot P(s)$$
$$= (0.2\cdot s)\cdot S(s) + (0.1\cdot s)\cdot L(s) - 0.05\cdot Kc\cdot(10\cdot s+1)\cdot P_{set}(s)$$

Characteristic Equation:

$$s^2 + (.12 - .5\cdot Kc)\cdot s - 5.0\cdot 10^{-2}\cdot Kc = 0$$

Eigenvalues:

$$r_1(Kc) := \left(-6.0\cdot 10^{-2} + .25\cdot Kc\right) + 1.0\cdot 10^{-2}\cdot\sqrt{36. + 200.\cdot Kc + 625.\cdot Kc^2}$$

$$r_2(Kc) := \left(-6.0\cdot 10^{-2} + .25\cdot Kc\right) - 1.0\cdot 10^{-2}\cdot\sqrt{36. + 200.\cdot Kc + 625.\cdot Kc^2}$$

for roots to be complex, need Kc to satisfy

$$36. + 200.\cdot Kc + 625.\cdot Kc^2 < 0$$

but $Kc = -0.16 + 0.178\cdot i$ $\quad Kc = -0.16 - 0.178\cdot i$ for $\quad 36. + 200.\cdot Kc + 625.\cdot Kc^2 = 0$

this means the roots can not be complex for any real-valued Kc.

For $r_1(Kc)$ to become positive,

$$\left(-6.0\cdot 10^{-2} + .25\cdot Kc\right) + 1.0\cdot 10^{-2}\cdot\sqrt{36. + 200.\cdot Kc + 625.\cdot Kc^2} > 0$$

we need, Kc<0. Thus the range required for Kc is simply:

$$Kc < 0$$

Alternatively, one can use the Routh-Hurwitz array:

$$\begin{pmatrix} 1 & -0.05\cdot Kc \\ 0.12 - 0.5\cdot Kc & 0 \\ -0.05\cdot Kc & 0 \end{pmatrix}$$

In order for the first column to have no sign change, we need:

$$Kc < 0$$

2. $$T(s) = \frac{0.4}{s + 0.32}\cdot T_{in}(s) + \left[0.8\cdot\left(\frac{1}{s + 0.32}\right)\cdot\left(\frac{8}{s + 1}\right)\right]\cdot x_v(s)$$

a) so the transfer function from x_v to T is given by,

$$\left[0.8\cdot\left(\frac{1}{s + 0.32}\right)\cdot\left(\frac{8}{s + 1}\right)\right] = \frac{6.4}{s^2 + 1.32\cdot s + .32}$$

b) so the transfer function from T_{in} to T is given by,

$$\frac{0.4}{s + 0.32}$$

c) using P-control, $\quad x_v(s) = Kc\cdot\left(T_{set}(s) - T(s)\right)$

$$T(s) = \left[\frac{0.4\cdot(s + 1)}{s^2 + 1.32\cdot s + .32 + 6.4\cdot Kc}\right]\cdot T_{in} + \left(\frac{6.4\cdot Kc}{s^2 + 1.32\cdot s + .32 + 6.4\cdot Kc}\right)\cdot T_{set}$$

Characteristic Equation: $s^2 + 1.32 \cdot s + .32 + 6.4 \cdot Kc = 0$

Eigengvalues: $r_1(Kc) := -.66 + 2.0 \cdot 10^{-2} \cdot \sqrt{289. - 16000. \cdot Kc}$

$r_2(Kc) := -.66 - 2.0 \cdot 10^{-2} \cdot \sqrt{289. - 16000. \cdot Kc}$

for the 45º in the left half complex plane, | real part | = | imaginary part |,

$$0.66 = 2.0 \cdot 10^{-2} \cdot \sqrt{(-289. + 16000. \cdot Kc)}$$

$$Kc = 8.6125 \cdot 10^{-2}$$

so for the roots to be inside the 45º region,

$$-5.0 \cdot 10^{-2} < Kc < 8.6125 \cdot 10^{-2}$$

d) at steady state,

$$T(\infty) = \lim_{s \to 0} s \cdot \left[\left[\frac{0.4 \cdot (s+1)}{s^2 + 1.32 \cdot s + .32 + 6.4 \cdot Kc} \right] \cdot \frac{0.5}{s} + \left(\frac{6.4 \cdot Kc}{s^2 + 1.32 \cdot s + .32 + 6.4 \cdot Kc} \right) \cdot 0.3 \cdot \frac{e^{-2 \cdot s}}{s} \right]$$

$$= \frac{(5. + 48. \cdot Kc)}{(8. + 160. \cdot Kc)}$$

$$offset = 0.3 - \frac{(5. + 48. \cdot Kc)}{(8. + 160. \cdot Kc)} = \frac{-.325}{(1. + 20. \cdot Kc)}$$

3. $G_P = \frac{0.2 \cdot (1 - 5 \cdot s)}{(10 \cdot s + 1) \cdot (3 \cdot s + 1)}$

$$G_V = \frac{8}{1.5 \cdot s + 1}$$

$$G_{CL} = \frac{K \cdot G_V \cdot G_P}{1 + K \cdot G_V \cdot G_P} = \frac{(1.6 - 8.0 \cdot s) \cdot K}{45.0 \cdot s^3 + 49.5 \cdot s^2 + (-8.0 \cdot K + 14.5) \cdot s + (1.6 \cdot K + 1.)}$$

Routh-Hurwitz Array:

$$\begin{bmatrix} 45 & -8 \cdot K + 14.5 \\ 49.5 & 1.6 \cdot K + 1 \\ -9.454 \cdot K + 13.59 & 0 \\ 1.6 \cdot K + 1 & 0 \end{bmatrix}$$

so for stability: $-0.625 < K < 1.437$

at K=1.437, two roots are pure imaginary, thus Ku=1.437

Using Ziegler Nichols rule, $K = 0.719$

$$offset = \lim_{s \to 0} s \cdot \left[1 - \frac{(1.6 - 8.0 \cdot s) \cdot K}{45.0 \cdot s^3 + 49.5 \cdot s^2 + (-8.0 \cdot K + 14.5) \cdot s + (1.6 \cdot K + 1.)}\right] \cdot \frac{\alpha}{s}$$

$$= \lim_{s \to 0} s \cdot \left[1 - \frac{(1.6 - 8.0 \cdot s) \cdot (0.719)}{45.0 \cdot s^3 + 49.5 \cdot s^2 + (-8.0 \cdot 0.719 + 14.5) \cdot s + (1.6 \cdot 0.719 + 1.)}\right] \cdot \frac{\alpha}{s}$$

$$= .465 \cdot \alpha$$

a) $\%offset = 46.5\%$

b) $$G_{CL,INNER} = \frac{K_1 \cdot G_V}{1 + K_1 \cdot G_V}$$

$$G_{CL,OUTER} = \frac{K_c \cdot G_{CL,INNER} \cdot G_P}{1 + K_c \cdot G_{CL,INNER} \cdot G_P}$$

$$= \frac{K_c \cdot \left(\frac{K_1 \cdot G_V}{1 + K_1 \cdot G_V}\right) \cdot G_P}{1 + K_c \cdot \left(\frac{K_1 \cdot G_V}{1 + K_1 \cdot G_V}\right) \cdot G_P} = \frac{K_c \cdot K_1 \cdot G_V \cdot G_P}{\left(1 + K_1 \cdot G_V + K_c \cdot K_1 \cdot G_V \cdot G_P\right)}$$

$$= \frac{\left(4.8 \cdot K_c - 24.0 \cdot K_c \cdot s\right)}{\left[45. \cdot s^3 + 769.5 \cdot s^2 + \left(-24. \cdot K_c + 326.5\right) \cdot s + \left(4.8 \cdot K_c + 25.\right)\right]}$$

Routh-Hurwitz Array:

$$\begin{bmatrix} 45 & \left(-24. \cdot K_c + 326.5\right) \\ 769.5 & \left(4.8 \cdot K_c + 25.\right) \\ -24.28 \cdot K_c + 325.0 & 0 \\ \left(4.8 \cdot K_c + 25.\right) & 0 \end{bmatrix}$$

$$K_u = 13.38$$

$$K_c = \frac{13.38}{2} = 6.69$$

$$\text{offset} = \lim_{s \to 0} s \cdot \left[1 - \frac{\left(4.8 \cdot K_c - 24.0 \cdot K_c \cdot s\right)}{\left[45. \cdot s^3 + 769.5 \cdot s^2 + \left(-24. \cdot K_c + 326.5\right) \cdot s + \left(4.8 \cdot K_c + 25.\right)\right]} \right] \cdot \left(\frac{\alpha}{s}\right)$$

$$. = \lim_{s \to 0} s \cdot \left[1 - \frac{(4.8 - 24.0 \cdot s) \cdot 6.69}{\left[45. \cdot s^3 + 769.5 \cdot s^2 + (-24. \cdot 6.69 + 326.5) \cdot s + (4.8 \cdot 6.69 + 25.)\right]} \right] \cdot \left(\frac{\alpha}{s}\right)$$

$$. = .438 \cdot \alpha$$

$$\%\text{offset} = 43.8\%$$