Linearization Examples

Example 1 : $\frac{dT}{dt} = T^2 - 80 \cdot T + 1500 = (T - 30) \cdot (T - 50)$

$f(T) := T^2 - 80 \cdot T + 1500$

$\Rightarrow$ There are two steady states: $T_{ss,1} = 30$ $\quad T_{ss,2} = 50$

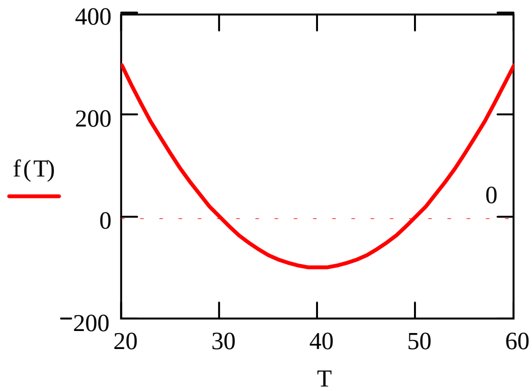
a) Linearizing around $T_{ss,1}$=30,

$$\frac{dT}{dt} = \left(2 \cdot T_{ss,1} - 80\right) \cdot \left(T - T_{ss,1}\right) = -20 \cdot T + 600$$

b) Linearizing around $T_{ss,2}$=50,

$$\frac{dT}{dt} = \left(2 \cdot T_{ss,2} - 80\right) \cdot \left(T - T_{ss,2}\right) = 20 \cdot T - 1000$$

$T := 20, 21 .. 60$



Remarks:

1. There can be more than one steady state.
2. Linearized equations are not used to determine steady states.

Let us look closer at the process: $\frac{dT}{dt} = T^2 - 80\cdot T + 1500$

We could solve this analytically since it is separable variables type.

$$\int \frac{1}{\left(T^2 - 80\cdot T + 1500\right)}\, dT = \int 1\, dt$$

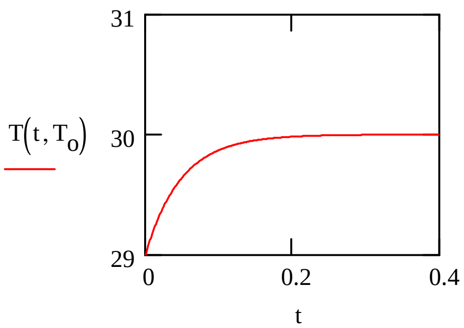
$$\frac{-1}{20}\cdot \ln(T - 30) + \frac{1}{20}\cdot \ln(T - 50) = t + C$$

$$T(t) = 10\cdot \frac{(-5 + 3\cdot \exp(20\cdot t + 20\cdot C))}{(\exp(20\cdot t + 20\cdot C) - 1)}$$

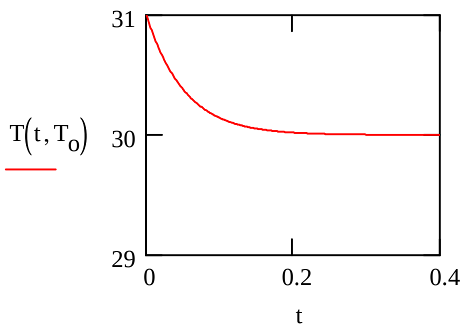
solving for C using initial condtion, T(0)=T_o , $C = \frac{1}{20}\cdot \ln\left[\frac{\left(-50 + T_o\right)}{\left(-30 + T_o\right)}\right]$

so finally we have, $T\left(t, T_o\right) := 10\cdot \frac{\left[\left(-150 + 3\cdot T_o\right)\cdot \exp(20\cdot t) + 150 - 5\cdot T_o\right]}{\left[\left(-50 + T_o\right)\cdot \exp(20\cdot t) + 30 - T_o\right]}$

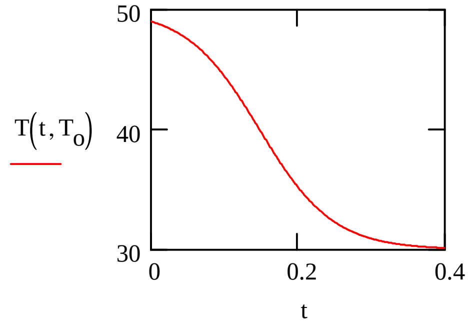
$t := 0, 0.001 .. 0.4$ $T_o := 29$



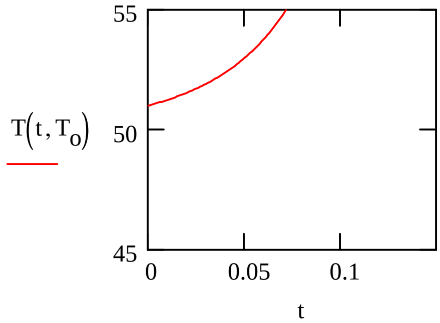
$T_o := 31$



$T_o := 49$

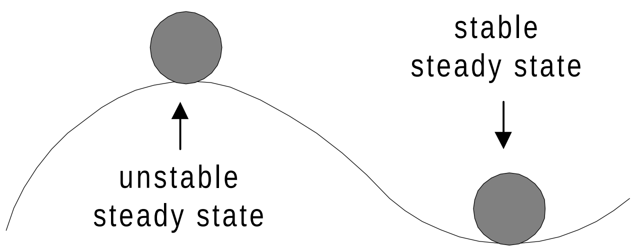


$T_o := 51$



Conclusion:

Even though T=30 and T=50 are both steady states, only T=30 is the stable steady state while T=50 is unstable steady state. This is similar to the situation where a boulder is either at the summit or a valley as shown in the figure below.



If we look back at the linearized equations, we note the following:

a) Around $T_{ss,1}$=30, $\frac{dT}{dt} + 20 \cdot T = 600$ Eigenvalues: s = -20

a) Around $T_{ss,2}$=50, $\frac{dT}{dt} - 20 \cdot T = -1000$ Eigenvalues: s = 20

Remark:

The linearized equations can still be used to determine the stability of a steady state. As long as the eigenvalues of the linearized equations (around a steady state) have negative real parts then that steady state is stable.

(Caution: if any of the eigenvalues turn out to be pure imaginary, then stability can not be determined using linearization method.)

Example 2 :

$$\frac{dh}{dt} = \frac{F_{in} - u\cdot\sqrt{h}}{10\cdot h^2} = f\left(h, u, F_{in}\right)$$

Linearize this around the operating points: $F_{in\_op} := 1$ $\quad h_{op} := 2$ $\quad u_{op} := 0.5$

$$\frac{dh}{dt} = \alpha + \beta\cdot\left(h - h_{op}\right) + \gamma\cdot\left(u - u_{op}\right) + \delta\cdot\left(F_{in} - F_{in\_op}\right)$$

$$\alpha = f\left(h_{op}, u_{op}, F_{in\_op}\right) = \frac{F_{in\_op} - u_{op}\cdot\sqrt{h_{op}}}{10\cdot h_{op}^{2}} = \frac{1 - 0.5\cdot\sqrt{2}}{10\cdot 2^2} = 7.322\times 10^{-3}$$

$$\beta = \frac{\partial}{\partial h} f\left(h_{op}, u_{op}, F_{in\_op}\right) = \left(\frac{-1}{20}\cdot\frac{u_{op}}{h_{op}^{\frac{5}{2}}}\right) - \left[\frac{1}{5}\cdot\frac{\left(F_{in\_op} - u_{op}\cdot h_{op}^{\frac{1}{2}}\right)}{h_{op}^{3}}\right] = -0.012$$

$$\gamma = \frac{\partial}{\partial u} f\left(h_{op}, u_{op}, F_{in\_op}\right) = \frac{-1}{10}\cdot h_{op}^{\frac{3}{2}} = -0.035$$

$$\delta = \frac{\partial}{\partial F_{in}} f\left(h_{op}, u_{op}, F_{in\_op}\right) = \frac{1}{10\cdot h_{op}^{2}} = 0.025$$

so finally, the linearized equation is given by,

$$\frac{dh}{dt} = 7.322\times 10^{-3} - 0.012\cdot(h - 2) - 0.035\cdot(u - 0.5) + 0.025\cdot\left(F_{in} - 1\right)$$

Remark: the coefficients show the local sensitivity of dh/dt to the different variables, i.e. the sign and magnitude show how each variable affect dh/dt .

Example 3: linearize the following equations around the steady states:

$$\frac{dP}{dt} = \left(-4\cdot\sqrt{P} + 5\cdot T\right) = f_1(T,P)$$

$$\frac{dT}{dt} = (T-30)\cdot(T-50) - 5\cdot P = f_2(T,P)$$

Solving for steady states,

$$0 = -4\cdot\sqrt{P_{ss}} + 5\cdot T_{ss}$$

$$0 = \left(T_{ss}-30\right)\cdot\left(T_{ss}-50\right) - 5\cdot P_{ss}$$

$$P_{ss} = 158.964 \qquad T_{ss} = 10.086$$

Linearized equations:

$$\frac{dP}{dt} = \alpha_1 + \beta_1\cdot\left(P - P_{ss}\right) + \gamma_1\cdot\left(T - T_{ss}\right)$$

$$\frac{dT}{dt} = \alpha_2 + \beta_2\cdot\left(P - P_{ss}\right) + \gamma_2\cdot\left(T - T_{ss}\right)$$

$$\alpha_1 = -4\cdot\sqrt{P_{ss}} + 5\cdot T_{ss} = 0$$

$$\alpha_2 = \left(T_{ss}-30\right)\cdot\left(T_{ss}-50\right) - 5\cdot P_{ss} = 0$$

$$\beta_1 = \frac{\partial}{\partial P}f_1\left(T_{ss},P_{ss}\right) = -0.159$$

$$\beta_2 = \frac{\partial}{\partial P}f_2\left(T_{ss},P_{ss}\right) = -5$$

$$\gamma_1 = \frac{\partial}{\partial T}f_1\left(T_{ss},P_{ss}\right) = 5$$

$$\gamma_2 = \frac{\partial}{\partial T}f_2\left(T_{ss},P_{ss}\right) = -59.83$$

Thus,

$$\frac{dP}{dt} = -0.159\cdot(P-15.9) + 5\cdot(T-10.086)$$

$$\frac{dT}{dt} = -5\cdot(P-15.9) - 59.83\cdot(T-10.086)$$

Example 4 : Linearize the following system around its steady state

$$\frac{d^2}{dt^2}P + P\cdot\left(\frac{d}{dt}P\right) + P^2 = 2$$

First introduce auxiliary variables, $y = \frac{dP}{dt}$

Then we have,

$$\frac{dP}{dt} = y = f_1(P, y)$$

$$\frac{dy}{dt} = \left(2 - P^2 - P\cdot y\right) = f_2(P, y)$$

Solve for steady states: $y_{SS} = 0$ $P_{SS} = \sqrt{2}$

Using the same method as in example 3, we arrive at the following:

$$\frac{dP}{dt} = y$$

$$\frac{dy}{dt} = -\sqrt{2}\cdot y - 2\cdot\sqrt{2}\cdot\left(P - \sqrt{2}\right)$$

Now combine both equations to get rid of y,

$$\frac{d^2}{dt^2}P + \sqrt{2}\cdot\left(\frac{d}{dt}P\right) + 2\cdot\sqrt{2}\cdot P = 4$$