Solution to ODEs

1. $\frac{d}{dt}x + a\cdot x = 3\cdot \exp(-2\cdot t)$ $\qquad x(0) = 2$

Characteristic equation: $s + a = 0$

Eigenvalue: $s = -a$

Complementary solution: $x_c = A\cdot \exp(-a\cdot t)$ [A is an arbitrary constant]

Particular solution: Case 1: $a \neq 2$

$x_p = B\cdot \exp(-2\cdot t)$

plug into ODE,

$-2\cdot B\cdot \exp(-2\cdot t) + a\cdot B\cdot \exp(-2\cdot t) = 3\cdot \exp(-2\cdot t)$

$B = \frac{3}{a-2}$

thus, $x_p = \left(\frac{3}{a-2}\right)\cdot \exp(-2\cdot t)$

Case 2: $a = 2$

$x_p = B\cdot t\cdot \exp(-2\cdot t)$

plug into ODE,

$(B\cdot \exp(-2\cdot t) - 2\cdot B\cdot t\cdot \exp(-2\cdot t)) + 2\cdot (B\cdot t\cdot \exp(-2\cdot t)) = 3\cdot \exp(-2\cdot t)$

$B = 3$

thus, $x_p = 3\cdot t\cdot \exp(-2\cdot t)$

complete solution:

Case 1: $a \neq 2$

$x(t) = A\cdot \exp(-a\cdot t) + \left(\frac{3}{a-2}\right)\cdot \exp(-2\cdot t)$

applying initial condition: $x(0) = 2$

$2 = A + \left(\frac{3}{a-2}\right)$ $\qquad$ or $\qquad$ $A = 2 - \frac{3}{(a-2)}$

thus,

$$x(t) = \left[2 - \frac{3}{(a-2)}\right]\cdot\exp(-a\cdot t) + \left[\frac{3}{(a-2)}\right]\cdot\exp(-2\cdot t)$$

Case 2: $a = 2$

$$x(t) = A\cdot\exp(-2\cdot t) + 3\cdot t\cdot\exp(-2\cdot t)$$

applying initial condition: $x(0) = 2$

$A = 2$

thus,

$$x(t) = (2 + 3\cdot t)\cdot\exp(-2\cdot t)$$

In summary,

case 1: $a \neq 2$ $\quad x(t) = \left[2 - \frac{3}{(a-2)}\right]\cdot\exp(-a\cdot t) + \left[\frac{3}{(a-2)}\right]\cdot\exp(-2\cdot t)$

case 2: $a = 2$ $\quad x(t) = (2 + 3\cdot t)\cdot\exp(-2\cdot t)$

steady state: if $a < 0$ no steady state

otherwise, at $t = \infty$ $\quad x_{steady\_state} = 0$

stability: if a=0, critically stable
if a>0, stable

2. $\frac{d^2}{dt^2}y + 5\cdot y = -3\cdot\frac{d}{dt}y + 2$ $\qquad y(0) = 1$ $\qquad \left(\frac{dy}{dt}\right)_{t=0} = \frac{1}{2}$

$$\frac{d^2}{dt^2}y + 3\cdot\frac{d}{dt}y + 5\cdot y = 2$$

Characteristic equation: $s^2 + 3\cdot s + 5 = 0$

Eigenvalue: $s = \begin{bmatrix} \frac{-3}{2} + \frac{1}{2}\cdot i\cdot\sqrt{11} \\ \frac{-3}{2} - \frac{1}{2}\cdot i\cdot\sqrt{11} \end{bmatrix}$

complementary solution: $y_c = \exp\left(-\frac{3}{2}\cdot t\right)\cdot\left(A\cdot\cos\left(\frac{\sqrt{11}}{2}\cdot t\right) + B\cdot\sin\left(\frac{\sqrt{11}}{2}\cdot t\right)\right)$

particular solution: $y_p = C$

plugging into ODE: $0 + 3\cdot 0 + 5\cdot C = 2$

$y_p = \frac{2}{5}$

Complete solution: $y = \exp\left(-\frac{3}{2}\cdot t\right)\cdot\left(A\cdot\cos\left(\frac{\sqrt{11}}{2}\cdot t\right) + B\cdot\sin\left(\frac{\sqrt{11}}{2}\cdot t\right)\right) + \frac{2}{5}$

Applying initial conditions:

$y(0) = 1 \qquad 1 = A + \frac{2}{5} \qquad \text{or} \qquad A = \frac{3}{5}$

$$\frac{dy}{dt} = \exp\left(-\frac{3}{2}\cdot t\right)\cdot\left[\left(\frac{-3}{2}\cdot B - \frac{1}{2}\cdot A\cdot\sqrt{11}\right)\cdot\sin\left(\frac{1}{2}\cdot\sqrt{11}\cdot t\right) + \left(\frac{-3}{2}\cdot A + \frac{1}{2}\cdot B\cdot\sqrt{11}\right)\cdot\cos\left(\frac{1}{2}\cdot\sqrt{11}\cdot t\right)\right]$$

$\left(\frac{dy}{dt}\right)_{t=0} = \frac{1}{2} = \frac{-3}{2}\cdot A + \frac{1}{2}\cdot B\cdot\sqrt{11} \qquad$ or $\qquad B = \frac{14}{55}\cdot\sqrt{11}$

$$y = \exp\left(\frac{-3}{2}\cdot t\right)\cdot\left(\frac{3}{5}\cdot\cos\left(\frac{1}{2}\cdot\sqrt{11}\cdot t\right) + \frac{14}{55}\cdot\sqrt{11}\cdot\sin\left(\frac{1}{2}\cdot\sqrt{11}\cdot t\right)\right) + \frac{2}{5}$$

Steady State: $y_{ss} = \frac{2}{5}$

stability : stable

3. $\frac{d}{dt}x = -3\cdot x + 2\cdot y + 3$ (eq 3.1)

$\frac{d}{dt}y + 5\cdot y = 2\cdot x - 4$ (eq 3.2)

Let $D = \frac{d}{dt}$ be the differential operator

$(D+3)\cdot x - 2\cdot y = 3$

$-2\cdot x + (D+5)\cdot y = -4$

in matrix form,

$$\begin{bmatrix} D+3 & -2 \\ -2 & D+5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} D+3 & -2 \\ -2 & D+5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D^2 + 8 \cdot D + 11} \cdot \begin{bmatrix} D+5 & 2 \\ 2 & D+3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D^2 + 8 \cdot D + 11} \cdot \begin{bmatrix} 7 \\ 6 - 12 \cdot a \end{bmatrix}$$

$$\frac{d^2}{dt^2} x + 8 \cdot \frac{d}{dt} x + 11 \cdot x = 7$$

characteristic equation: $s^2 + 8 \cdot s + 11 = 0$

eigenvalues: $s = \begin{bmatrix} -4 + \sqrt{5} \\ -4 - \sqrt{5} \end{bmatrix}$

complementary solution:

$$x_c = A \cdot \exp\left[\left(-4 + \sqrt{5}\right) \cdot t\right] + B \cdot \exp\left[\left(-4 - \sqrt{5}\right) \cdot t\right]$$

Particular solution:

$$x_p = \frac{7}{11}$$

complete solution:

$$x = \frac{7}{11} + \left[A \cdot \exp\left[\left(-4 + \sqrt{5}\right) \cdot t\right] + B \cdot \exp\left[\left(-4 - \sqrt{5}\right) \cdot t\right]\right]$$

To obtain y, it is simpler to use eqn 3.1

$$\frac{d}{dt} x = -3 \cdot x + 2 \cdot y + 3$$

$$y = \frac{1}{2}\cdot\left(\frac{d}{dt}x + 3\cdot x - 3\right)$$

$$y = \left[\frac{1}{2}\cdot\left(-1+\sqrt{5}\right)\cdot A\right]\cdot\exp\left[\left(-4+\sqrt{5}\right)\cdot t\right] + \left[\frac{-1}{2}\cdot\left(\sqrt{5}+1\right)\cdot B\right]\cdot\exp\left[\left(-4-\sqrt{5}\right)\cdot t\right] - \frac{6}{11}$$

Applying initial conditions,

$x(0) = 1 \qquad 1 = \frac{7}{11} + A + B$

$y(0) = -1 \qquad -1 = \left(\frac{-1}{2} + \frac{1}{2}\cdot\sqrt{5}\right)\cdot A + \left(\frac{-1}{2}\cdot\sqrt{5} - \frac{1}{2}\right)\cdot B - \frac{6}{11}$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{2}{11} - \frac{3}{55}\cdot\sqrt{5} \\ \frac{3}{55}\cdot\sqrt{5} + \frac{2}{11} \end{bmatrix}$$

$$x = \frac{7}{11} + \left(\frac{2}{11} - \frac{3}{55}\cdot\sqrt{5}\right)\cdot\exp\left[\left(-4+\sqrt{5}\right)\cdot t\right] + \left(\frac{3}{55}\cdot\sqrt{5} + \frac{2}{11}\right)\cdot\exp\left[\left(-4-\sqrt{5}\right)\cdot t\right]$$

$$y = \left(\frac{-5}{22} + \frac{13}{110}\cdot\sqrt{5}\right)\cdot\exp\left[\left(-4+\sqrt{5}\right)\cdot t\right] + \left(\frac{-5}{22} - \frac{13}{110}\cdot\sqrt{5}\right)\cdot\exp\left[\left(-4-\sqrt{5}\right)\cdot t\right] - \frac{6}{11}$$

Steady state: $\begin{bmatrix} x_{ss} \\ y_{ss} \end{bmatrix} = \begin{bmatrix} \frac{7}{11} \\ -\frac{6}{11} \end{bmatrix}$

Stability: stable

4. $\frac{d^2}{dt^2}z + 4\cdot z = \cos(2\cdot t) \qquad z(0) = 1 \qquad \left(\frac{dz}{dt}\right)_{t=0} = 0$

characteristic equation: $s^2 + 4 = 0$

eigenvalues: $\begin{bmatrix} 2i \\ -2\cdot i \end{bmatrix}$

Complementary solution: $z_c = A\cdot\cos(2\cdot t) + B\cdot\sin(2\cdot t)$

Particular solution: $z_p = t \cdot (C \cdot \cos(2 \cdot t) + D \cdot \sin(2 \cdot t))$

plugging into equation,

$$-4 \cdot C \cdot \sin(2 \cdot t) + 4 \cdot D \cdot \cos(2 \cdot t) = \cos(2 \cdot t)$$

$$C = 0 \qquad D = \frac{1}{4}$$

Complete solution:

$$z = \frac{1}{4} \cdot t \cdot \sin(2 \cdot t) + (A \cdot \cos(2 \cdot t) + B \cdot \sin(2 \cdot t))$$

Applying initial conditions:

$$1 = A$$

$$0 = 2 \cdot B$$

$$z = \frac{1}{4} \cdot t \cdot \sin(2 \cdot t) + \cos(2 \cdot t)$$

Steady state: not applicable

Stability: bounded sustained oscillation, but not asymptotically stable