P, PI, PID Control:

1. Given: $5\cdot\frac{dx_1}{dt}=-x_1+2\cdot u$

$2\cdot\frac{dx_2}{dt}=-x_2+u$

$y=x_1-x_2$

using P-control on y: $u=k_c\cdot(y_{set}-y)$ $u_o=0$

substituting, $5\cdot\frac{dx_1}{dt}=-x_1+2\cdot k_c\cdot(y_{set}-x_1+x_2)$

$2\cdot\frac{dx_2}{dt}=-x_2+k_c\cdot(y_{set}-x_1+x_2)$

rearranging, $\frac{dx_1}{dt}=\frac{-1-2\cdot k_c}{5}\cdot x_1+\frac{2\cdot k_c}{5}\cdot x_2+\frac{2\cdot k_c}{5}\cdot y_{set}$

$\frac{dx_2}{dt}=\frac{-k_c}{2}\cdot x_1+\frac{-1+k_c}{2}\cdot x_2+\frac{k_c}{2}\cdot y_{set}$

characteristic equation:

$$s^2-\left(\frac{-7}{10}+\frac{1}{10}\cdot k_c\right)\cdot s+\left(\frac{1}{10}+\frac{1}{10}\cdot k_c\right)=0$$

eigenvalues will be stable if: $-1<k_c<7$

2. Given: $-10\cdot\frac{dx}{dt}+x=\frac{1}{5}\cdot u$

Using PI-control, $u=k_c\cdot\left[(x_{set}-x)+\frac{1}{\tau_I}\cdot\int(x_{set}-x)\,dt\right]$ $(u_o=0)$

$$-10\cdot\frac{dx}{dt}+x=\frac{1}{5}\cdot k_c\cdot\left[(x_{set}-x)+\frac{1}{\tau_I}\cdot\int(x_{set}-x)\,dt\right]$$

$$-10\cdot\frac{d^2}{dt^2}x+\frac{dx}{dt}=\frac{k_c}{5}\cdot\left[-\frac{dx}{dt}+\frac{1}{\tau_I}\cdot(x_{set}-x)\right]$$

$$-10\cdot\frac{d^2}{dt^2}x+\left(1+\frac{k_c}{5}\right)\cdot\frac{dx}{dt}+\frac{k_c}{5\cdot\tau_I}\cdot x=\frac{k_c}{5\cdot\tau_I}\cdot x_{set}$$

$$\frac{-50\cdot\tau_I}{k_c}\cdot\left(\frac{d^2}{dt^2}\,x\right)+\left(\frac{5+k_c}{k_c}\cdot\tau_I\right)\cdot\frac{dx}{dt}+x=x_{set}$$

For a decay ratio = 1/3, we need the damping coefficient to satisfy:

$$\exp\left(\frac{-2\cdot\pi\cdot\zeta}{\sqrt{1-\zeta^2}}\right)=\frac{1}{4}$$

$$\zeta:=\frac{\ln(4)}{\sqrt{\left(\ln(4)^2+4\cdot\pi^2\right)}}\qquad \zeta=0.215$$

For a period of oscillation, $T:=10$

$$\omega=\frac{2\cdot\pi}{T}=\frac{\sqrt{1-\zeta^2}}{\tau_n}$$

$$\tau_n:=\frac{1}{2}\cdot\sqrt{\left(1-\zeta^2\right)}\cdot\frac{T}{\pi}\qquad \tau_n=1.554$$

Thus, comparing coefficients, we have

$$\tau_n^2=\frac{-50\cdot\tau_I}{k_c}\qquad \text{or}\qquad k_c=-50\cdot\frac{\tau_I}{\tau_n^2}$$

$$2\cdot\zeta\cdot\tau_n=\frac{5+k_c}{k_c}\cdot\tau_I=\frac{-1}{50}\cdot\left(5-50\cdot\frac{\tau_I}{\tau_n^2}\right)\cdot\tau_n^2=\frac{-1}{10}\cdot\tau_n^2+\tau_I$$

$$\tau_I:=2\cdot\zeta\cdot\tau_n+\frac{1}{10}\cdot\tau_n^2\qquad \tau_I=0.911$$

$$k_c:=-50\cdot\frac{\tau_I}{\tau_n^2}\qquad k_c=-18.863$$